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DYNAMIC COMPOSITE LAMINATE FINITE ELEMENT ANALYSIS.(U)
MAR 81 J O'CALLAHAN, J A MCELMAN F33615-78-C-2052
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DYNAMIC COMPOSITE LAMINATE FINITE ELEMENT ANALYSIS

University of Lowell
College of Engineering
Lowell, Massachusetts

March 1981

TECHNICAL REPORT AFWAL-TR-80-2027
Final Report for Period August 1978 - November 1979

AERO PROPULSION LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
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This technical report has been reviewed and is approved for publication.

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: AFWAL-TR-80-2027

2. GOVT ACCESSION NO.: AD-A100643

3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): Dynamic Composite Laminate Finite Element Analysis

5. TYPE OF REPORT & PERIOD COVERED: Technical - Final, August 1978-November 1979

6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): Dr. John O'Callahan & Dr. John A. McElman

8. CONTRACT OR GRANT NUMBER(s): F33615-78-C-2052

9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Lowell, College of Engineering, Lowell, Massachusetts

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 3066/12/53

11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Wright Aeronautical Laboratories, Attn: AFWAL/POTP (Theodore G. Fecke), Wright-Patterson Air Force Base, Ohio 45433

12. REPORT DATE: March 1981

13. NUMBER OF PAGES:

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):

15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

SAP IV, Composite Material, Composite Blades, Finite Element, Analysis, Laminate Theory, Plate Element, F.O.D., Composite Analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The analysis of plate like structures such as blades built-up with composite laminate fibers requires the modification of an existing finite element computer program to include the coupling of in-plane stretching with out-of-plane bending of a plate. An industry standard computer program SAP IV was selected as host program to accept new composite plate finite element.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Block 20 ABSTRACT (Cont'd)

The SAP IV Finite Element Computer Program was designed to accept new elements into its element library easily. The new element, in general, must be self contained since the general philosophy and program structure is "overlayed" into the computer. The laminate composite plate element is the new element to be integrated into the element library. The new element named TYPE 9 is similar to element TYPE 6 in element description and input. The main difference is that element TYPE 9 has the ability to couple in-plane extension with out-of-plane bending which is possible with laminate plate behavior and theory. Element TYPE 9 is a quadrilateral element and is formulated from quadrilateral shape functions rather than by four triangles as in TYPE 6. Also, in general, TYPE 9 allows for material directions to be arbitrary for ease of material input descriptions. The element is modelled after the structure of element TYPE 6; therefore, element TYPE 9 can degenerate to element TYPE 6.

FOREWORD

The work described herein was sponsored by the Department of the Air Force, AFSC Wright-Patterson AFB under contract number F33615-78-C-2052 for the period of August 1978 to November 1979. The work was performed at the University of Lowell, Lowell, Massachusetts by Drs. John C. O'Callahan and John A. McElman as co-principal investigator with assistance from Dr. G. Dudley Shepard of the Mechanical Engineering Department and Mr. Wen Wei Shui, a graduate research assistant. The work was coordinated by Mr. Ted G. Fecke of AFWAL/POTP of Wright-Patterson and Capt. Paul Copp of the Air Force Academy.

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LIST OF SYMBOLS

SYMBOL	DESCRIPTION
$\underset{\sim}{\alpha}$	VECTOR OF THERMAL EXPANSION COEFFICIENTS
a_X, a_Y, a_Z	ACCELERATION COEFFICIENTS IN X,Y,Z DIRECTIONS
$\underset{\sim}{b}$	VECTOR OF ELEMENT BODY FORCES
$\underset{\sim}{\beta}$	VECTOR OF GENERALIZED COEFFICIENTS
$\underset{\sim}{B}_o$	THE TRANSVERSE STRAIN-DISPLACEMENTS RELATIVE TO $\underset{\sim}{q}_o$
$\underset{\sim}{B}_I$	THE IN-PLANE STRAIN-DISPLACEMENTS RELATIVE TO $\underset{\sim}{q}_I$
$\underset{\sim}{\tilde{C}}$	MATERIAL MATRIX DESCRIBED IN THE PLATE LOCAL AXES
$\underset{\sim}{\partial}_n^i$	"i"th DERIVATIVE OPERATOR IN THE NATURAL REFERENCE FRAME
$\underset{\sim}{\partial}_l^i$	"i"th DERIVATIVE OPERATOR IN THE LOCAL REFERENCE FRAME
δ	FIRST VARIATIONAL OPERATOR
$\underset{\sim}{\varepsilon}$	VECTOR OF STRAIN COMPONENTS CORRESPONDING TO $\underset{\sim}{\sigma}$
$\underset{\sim}{\varepsilon}_o$	MID-PLANE STRAIN VECTOR
$\underset{\sim}{\varepsilon}_t$	THERMAL STRAIN VECTOR
$\underset{\sim}{\bar{\varepsilon}}$	VECTOR OF NEW (NATURAL) STRAIN COMPONENTS
$\underset{\sim}{E}$	MODULUS OF ELASTICITY MATRIX
$\hat{e}_{yi}, \hat{e}_{xi}$	ELEMENT'S LOCAL y AND x COORDINATES AT NODE "i"
$\hat{e}_{zi}$	ELEMENT NORMAL COORDINATES AT NODE "i"
$\underset{\sim}{E}_m$	MATERIAL MATRIX
$\underset{\sim}{F}$	FORCE MATRIX
$\underset{\sim}{F}_g$	GLOBAL THERMAL VECTOR

LIST OF SYMBOLS (CONT'D)

SYMBOL	DESCRIPTION
$\underline{F}_n$	NATURAL THERMAL VECTOR
$\underline{F}_z^p$	PRESSURE LOAD VECTOR IN z DIRECTION
$\underline{F}_g^p$	GLOBAL PRESSURE LOAD VECTOR
$\underline{F}_g^a$	ACCELERATION VECTOR
f	INPUT SCALING FACTOR
$\tilde{G}$	SHEAR MODULUS MATRIX
γ	TRANSVERSE SHEAR DEFORMATION
$\underline{g}_n$	NATURAL ELEMENT STIFFNESS VECTOR
$\underline{g}_g$	GLOBAL ELEMENT STIFFNESS VECTOR
$\underline{H}$	VECTOR CONTAINING TERMS OF THE SHAPE FUNCTION OF AN ELEMENT
$\tilde{J}$	THE JACOBIAN MATRIX
$\underline{K}$	VECTOR OF PLATE CURVATURES
$\tilde{K}_n$	NATURAL STIFFNESS MATRIX
$\tilde{K}_{\theta z}$	ARTIFICIAL TORSIONAL STIFFNESS MATRIX RELATIVE TO $\underline{\theta}_z$
$\tilde{K}_g$	GLOBAL STIFFNESS MATRIX
L	TOTAL NUMBER OF FIBER LAMINA LEVELS
$\underline{M}$	MOMENT RESULTANTS VECTOR
$\tilde{M}^c$	MASS MATRIX RELATIVE TO IN-PLANE VARIABLES IN ONE COORDINATE
$\tilde{M}^l$	LUMPED MASS MATRIX
$\underline{M}_g^l$	GLOBAL LUMPED MASS VECTOR
$\underline{N}$	STRESS RESULTANTS VECTOR
$\underline{P}$	VECTOR OF SURFACE TRACTIONS APPLIED ON SURFACE
Π_p	SUM OF U_p AND V_p

LIST OF SYMBOLS (CONT'D)

SYMBOL	DESCRIPTION
$\underline{\psi}$	VECTOR OF POLYNOMIAL COEFFICIENTS
$\underset{\sim}{\Psi}$	THE MATRIX ϕ EVALUATED AT THE NODES, DEFINED IN REFERENCE 2
$\bar{\underline{q}}_n$	NEW (NATURAL) SET OF DEGREES OF FREEDOM
$\underline{q}$	NODAL DISPLACEMENTS VECTOR
$\underline{q}_{nL}$	VECTOR OF NATURAL COORDINATE VARIABLES, PER NODE
$\underline{q}_n$	NATURAL-TO-LOCAL TRANSFORMATION VECTOR
$\underline{q}_l$	LOCAL DEGREES OF FREEDOM FOR ELEMENT
$\underline{q}_g$	GLOBAL DEGREES OF FREEDOM FOR ELEMENT
(r,s)	ELEMENT NATURAL COORDINATES
$\underset{\sim}{R}_E$	STRAIN TRANSFORMATION MATRIX, FROM ELEMENT LOCAL COORDINATES TO PRINCIPAL FIBER DIRECTIONS
r_k, s_l	ROOTS OF LEGENDRE POLYNOMIAL
ρ	MASS DENSITY
$\bar{\underline{\Sigma}}$	COMBINED STRESS-STRAIN VECTOR
$\underline{\sigma}$	VECTOR OF STRESS COMPONENTS
$\bar{\underline{\sigma}}_t$	THERMAL STRESS VECTOR FORMED FROM STRAINS
$\underset{\sim}{S}$	STRESS MATRIX
$\underline{\phi}^T$	SHAPE FUNCTION POLYNOMIAL
$\underset{\sim}{t}_{nl}$	NATURAL-TO-LOCAL TRANSFORMATION MATRIX
$\underset{\sim}{t}_{Ri}$	ROTATIONAL TRANSFORMATION MATRIX AT NODE "i" IN NATURAL-TO-GLOBAL TRANSFORMATION
$\underset{\sim}{t}_{Di}$	DISPLACEMENT TRANSFORMATION MATRIX AT NODE "i" IN NATURAL-TO-GLOBAL TRANSFORMATION
t_i	LOCAL-TO-GLOBAL COORDINATE TRANSFORMATION MATRIX AT NODE "i"

LIST OF SYMBOLS (CONT'D)

SYMBOL	DESCRIPTION
$\underline{T}_{ngi}$	NODAL NATURAL-TO-GLOBAL STIFFNESS TRANSFORMATION MATRIX
$\underline{T}_{ng}$	NATURAL-TO-GLOBAL ELEMENT STIFFNESS TRANSFORMATION MATRIX
θ_z	THE ROTATIONAL DEGREE OF FREEDOM NORMAL TO PLATE AT A NODE
T_g	ELEMENT THERMAL GRADIENT THROUGH PLATE THICKNESS
T_o	ELEMENT MEAN TEMPERATURE DIFFERENCE
θ_x	DEGREE OF ROTATION ABOUT x-AXIS
θ_y	DEGREE OF ROTATION ABOUT y-AXIS
t	POSITION THROUGH THICKNESS OF LAMINATE
u	DISPLACEMENT IN x-DIRECTION
$\underline{u}$	A VECTOR OF ELEMENT DISPLACEMENTS
U_p	SUM OF STRAIN ENERGY
$\underline{U}$	UPPER TRI-DIAGONAL FACTORING MATRIX
$\underline{V}_m$	GLOBAL MATERIAL REFERENCE VECTOR
V_p	POTENTIAL OF ALL APPLIED LOADS
V	VOLUME OF AN ELEMENT
v	DISPLACEMENT IN y-DIRECTION
w	TRANSVERSE DISPLACEMENT
W_K, W_I	GAUSS WEIGHTING FACTORS
X,Y,Z	GLOBAL COORDINATES
x,y,z	LOCAL COORDINATES

SECTION I
INTRODUCTION

This report describes research which was directed at the development of an orthotropic plate finite element for the analysis of plate and shell-like structures which exhibit coupling between extension and bending. The element is especially useful in the analysis of structures which are fabricated from laminated composite materials. The report is written so that it would describe, for the finite element expert, the analytical techniques utilized in the development of the element and also would be of use to a program "user" not having the overall expertise of an expert.

The notation and the methods used for the definition of element material properties have been chosen as a result of a careful survey of the literature on composite materials. The notation and definitions chosen are considered to be industry standard and are best summarized in reference 3, which is rapidly becoming a standard text for the analysis of composite materials.

An industry standard computer program SAP IV [(1)] was selected as host program to accept the new composite plate finite element. The SAP IV Finite Element Computer Program was designed to easily accept new elements into its element library. The new element must be self contained since the general philosophy and program structure is "overlayed" into the computer. The laminate composite plate element is the new element to be integrated into the element library. Called TYPE 9, the new element is similar to the SAP element TYPE 6 in both description and input. The main difference is that element TYPE 9 has the ability to describe the effects of coupling between in-plane extension and out-of-plane bending. Element TYPE 9 is a quadrilateral element and is formulated from quadrilateral shape functions rather than from four triangles as in TYPE 6. Also, TYPE 9 allows material directions

to be arbitrary for ease of material input descriptions. The element is modelled after the structure of element TYPE 6; therefore element TYPE 9 can degenerate to element TYPE 6.

SECTION II
LAMINATE COMPOSITE FLAT PLATE ELEMENT

The laminate composite flat plate element is based on thin plate theory with the exclusion of transverse shear deformations. The following sections describe the basic formulation of the finite element.

1. ELEMENT POTENTIAL ENERGY FUNCTIONAL

The principle of minimum potential energy furnishes a variational basis for the direct formulation of the element stiffness equations and loading functions. The potential energy of the element is formed from the sum of strain energy (U_p) and the potential of all applied loads (V_p); i.e.,

$$\pi_p = U_p + V_p \tag{1}$$

The principle can be stated as follows: Among all the displacement functions of admissable form, those that satisfy the element equilibrium conditions make the potential energy functional obtain a stationary value. Thus,

$$\delta\pi_p = \delta U_p + \delta V_p = o \tag{2}$$

where δ is the first variational operator.
It can be shown that

$$\delta U_p = \int_V \underline{\sigma}^T \; \delta\underline{\varepsilon} \; dV \tag{3}$$

where $\underline{\sigma}^T$ is a vector of stress components,
$\underline{\varepsilon}$ the corresponding vector of strain components and
V the volume of the element.

Note: All vectors will be underscored with a straight bar and matrices will be underscored with a tilda. The superscript ^{T} of the vectors and matrices designates the matrix is transposed.

The corresponding first variation of the potential forces becomes

$$\delta V_p = -\int_V \underline{b}^T \, \delta\underline{u} \, dV - \int_S \underline{p}^T \, \delta\underline{u} \, dS \qquad (4)$$

where $\underline{b}$ represents the element body forces,

$\underline{p}$ is a vector of surface tractions applied on surface S, and

$\underline{u}$ is a vector of element displacements.

Note that the surface traction integral can be used to include the point concentrated forces on the boundaries of the element.

The elements of the strain potentials of equation (3) will eventually lead to the element stiffness and initial load vectors and the elements of the applied load potential of equation (4) will produce the various element vectors.

2. QUADRILATERAL SHAPE FUNCTIONS

The element formulation is a geometrically linear quadrilateral containing the four corner nodes as shown in FIGURE 1.

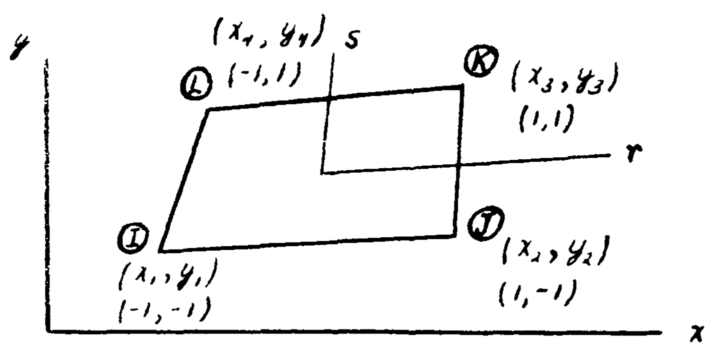


FIGURE 1 - Quadrilateral Element Geometry

a. Geometric Shape Functions

The element shown in FIGURE 1 is described in the local coordinate of the element and all material reference is made with respect to the element local x axis. The element area domain can be described by using a polynomial as

$$x = \underline{\phi}^T \underline{\beta} \tag{5}$$
$$y = \underline{\phi}^T \underline{\beta}$$

where x and y are the local coordinates as in the element domain,

$$\underline{\phi}^T = \lfloor 1, r, s, rs \rfloor \tag{6}$$

the row vector of polynomial coefficients $\underline{\beta}$, a vector of generalized coefficients, and r,s, the element natural coordinates.

The generalized coefficients can be solved for by evaluating the polynomials at the vertices of the element. Therefore,

$$x = \underline{H}^T \underline{x} \tag{7}$$
$$y = \underline{H}^T \underline{y}$$

where $\underline{H}^T$ contains the terms of the shape function of the element; $\underline{x}$ and $\underline{y}$ are vectors containing the element vertices as,

$$\underline{x}^T = \lfloor x_1, x_2, x_3, x_4 \rfloor \tag{8}$$
$$\underline{y}^T = \lfloor y_1, y_2, y_3, y_4 \rfloor$$

The terms of the shape function can be described by

$$h_i = \frac{1}{4} (1 + r_i r) (1 + s_i s) \tag{9}$$

where

$$\underline{r}_i^T = \lfloor -1, 1, 1, -1 \rfloor \tag{10}$$
$$\underline{s}_i^T = \lfloor -1, -1, 1, 1 \rfloor$$

define the natural coordinates of the elements.

The mapping of the element geometry and displacement functions can be obtained by defining the Jacobian transformation as

$$\underline{\partial}_n = \underset{\sim}{J}\ \underline{\partial}_\ell \tag{11}$$

where $\underset{\sim}{J}$ is the Jacobian matrix defined as

$$\underset{\sim}{J} = \begin{bmatrix} x,_r & y,_r \\ x,_s & y,_s \end{bmatrix} \tag{12}$$

and

$$\underline{\partial}_n = \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{Bmatrix}$$
$$\underline{\partial}_\ell = \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \tag{13}$$

are the first derivative operators in the natural and local reference frames respectively.

Note: The "," subscript implies "partial differentiation with respect to". The inverse transformation is obtained by

$$\underline{\partial}_\ell = \frac{1}{J^*}\ \underset{\sim}{G}\ \underline{\partial}_n \tag{15}$$

where J^* is the determinant of the Jacobian matrix given as

$$J^* = x,_r\ y,_s - x,_s\ y,_r \tag{16}$$

and,

$$\underset{\sim}{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} x,_s & -y,_r \\ -x,_s & y,_r \end{bmatrix} \tag{17}$$

It will be necessary to obtain second derivates in the local reference; therefore

$$\underline{\partial}_\ell^2 = \underset{\sim}{E}\, \underline{\partial}_n + \underset{\sim}{F}\, \underline{\partial}_n^2 \tag{18}$$

represents the second partial operator in the local reference given as

$$\underline{\partial}_\ell^2 = \begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} \tag{19}$$

and the natural set as

$$\underline{\partial}_n^2 = \begin{Bmatrix} \dfrac{\partial^2}{\partial r^2} \\ \dfrac{\partial^2}{\partial r \partial s} \\ \dfrac{\partial^2}{\partial s^2} \end{Bmatrix} \tag{20}$$

The $\underset{\sim}{E}$ matrix is defined as

$$\underset{\sim}{E} = \begin{bmatrix} \underline{e}_{11}^t \\ \underline{e}_{22}^t \\ \underline{e}_{12}^t \end{bmatrix} \tag{21}$$

where

$$\underline{e}_{ij}^{T} = \frac{1}{J^{*2}} \underline{g}_i^T \underline{\partial}_n^* \underline{g}_j^T \tag{22}$$

with $\underline{g}_i^T$ being the ith row partition out of the $\underset{\sim}{G}$ matrix and

$$\underline{\partial}_n^* = \underline{\partial}_n - \frac{1}{J^*} \underline{\partial}_n \; J^* \tag{23}$$

The $\underset{\sim}{F}$ matrix is defined as

$$\underset{\sim}{F} = \begin{bmatrix} \underline{f}_{11}^T \\ \underline{f}_{22}^T \\ \underline{f}_{12}^T \end{bmatrix} \tag{24}$$

where

$$\underline{f}_{ij}^T = \frac{1}{J^{*2}} \bar{\underline{g}}_i^T \; \bar{\underset{\sim}{g}}_j \tag{25}$$

with

$$\bar{\underset{\sim}{g}}_i = \begin{bmatrix} g_{i1} & g_{i2} & 0 \\ 0 & g_{i1} & g_{i2} \end{bmatrix} \tag{26}$$

the elements being of the $\underset{\sim}{G}$ matrix.

b. In-Plane Displacement Shape Function

The plate element is assumed to have in-plane deformations; therefore the variation of x and y displacements, u and v respectively, can be expressed using the same shape functions for the geometry as in the previous section. Then

$$\begin{aligned} u &= \underline{H}^T \; q_u \\ v &= \underline{H}^T \; q_v \end{aligned} \tag{27}$$

are the domain displacements of the element

$$\text{where} \quad \underline{q}_u^T = \lfloor u_1 \; u_2 \; u_3 \; u_4 \rfloor$$
$$\text{and} \quad \underline{q}_v^T = \lfloor v_1 \; v_2 \; v_3 \; v_4 \rfloor \tag{28}$$

contain the in-plane model displacements. Therefore, it is assumed that in-plane displacements vary linearly within the element.

c. Transverse Displacement Shape Function

The plate element defined by thin plate theory must have a transverse shape function to allow for proper bending. Therefore it is assumed that the shape function polynomial is

$$\underline{\phi}^T = \lfloor 1, r, s, r^2, rs, s^2, r^3, r^2s, rs^2, s^3, r^3s, rs^3 \rfloor \tag{29}$$

The natural degrees of freedom allowed per node for bending are

$$\underline{q}_{0i}^T = \lfloor w \quad w,_r \quad w,_s \rfloor_i \tag{30}$$

Letting

$$\underline{q}_0 = \underset{\sim}{\psi} \, \underline{\beta} \tag{31}$$

where $\underline{q}_0$ is the full set of degrees of freedom in the natural reference of the element, $\underset{\sim}{\psi}$ is the matrix $\underline{\Phi}$ evaluated at the nodes and defined in reference 2 and $\underline{\beta}$ is the set of generalized nodal coefficients;

Then the transverse displacement w becomes

$$w = \underline{\Phi}_n^T \, \underset{\sim}{\psi}^{-1} \, \underline{q}_0 = \underline{\bar{H}}^T \, \underline{q}_0 \tag{32}$$

where $\underset{\sim}{\psi}^{-1}$ is the inverse of $\underset{\sim}{\psi}$ and $\underline{\bar{H}}$ is the transverse shape function, both of which are defined in reference 2.

The transverse displacement shape functions are defined in the natural reference of the element to allow for ease of development since second derivatives must be taken. Since a global transformation by node is to be performed at a later stage, the transformation by node from the natural to local coordinate can be made.

3. PLATE STRAIN FUNCTIONS

The classical assumptions of linear thin plate theory are made, essentially reducing the three-dimensional equations of elacticity to a two-dimensional set of plane stress equations. For the elastic continuum of the plate, the following assumptions are made:

- The thickness (h) is small compared to the dimensions of the plate in the x and y directions.
- A line element through the thickness remains normal to the mid-plane surface under all states of deformation, independent of its translation or rotation.
- The plate can be isotropic, orthotropic or comprised of a number of orthotropic laminae, where each lamina obeys Hooke's law.
- The displacements u, v, and w in the x, y, and z directions respectively, are small when compared to the plate thickness.
- The reference axis is taken as the middle of the plate at h/2, h being the total plate thickness.
- The normal strain in the z-direction is assumed to be zero, giving

$$\varepsilon_z = w,_z = 0;$$

therefore, the lateral deflection is given by,

$$w = w(x,y) .$$

- St. Venant's principle applies. That is, local deformation occurs in the area of applied loads while at distances away from the load, the deformation state is not grossly affected.
- Transverse shear deformations are neglected;

$$\gamma_{xz} = \gamma_{yz} = 0 .$$

- Displacements are linear such that

$$u = u_0(x,y) - zw,_x \quad \text{and} \quad v = v_0(x,y) - zw,_y ,$$

where $w,_x = -\theta_y$, $w,_y = \theta_x$, and u_0 and v_0 are the in-plane displacements of the middle surface. The rotations about the x and y axes are given by θ_x and θ_y, respectively.

a. Midplane Strain and Curvatures

The mechanical strains associated with plate stretching and bending can be written as

$$\underset{-}{\varepsilon} = \underset{-}{\varepsilon}_o + z\,\underset{-}{\kappa} \tag{33}$$

where the mid-plane strains are

$$\underset{-}{\varepsilon}_o = \begin{Bmatrix} u,_x \\ v,_y \\ u,_y + v,_x \end{Bmatrix} , \tag{34}$$

the plate curvatures are

$$\underset{-}{\kappa} = \begin{Bmatrix} -w,_{xx} \\ -w,_{yy} \\ -2w,_{xy} \end{Bmatrix} \tag{35}$$

with u and v being the in-plane displacements and w, the transverse displacements. The thermal strains can be written as

$$\underline{\varepsilon}_t = \underline{\tilde{\alpha}}\, T_o + z\, \underline{\tilde{\alpha}}\, T_g \tag{36}$$

where $\underline{\tilde{\alpha}}$ is a vector of thermal expansion coefficients relative to mid-plane strains, T_o the element mean temperature difference and T_g the element thermal gradient through the plate thickness.

b. Strain Displacement Functions

The connection between strain and displacement is made realizing that the in-plane and transverse displacements have been made relative to a set of nodal displacements. Equation (33) can be written as

$$\underline{\varepsilon} = \underset{\sim}{B}_I\, q_I + z\underset{\sim}{B}_0\, q_0 \tag{37}$$

where $\underset{\sim}{B}_I$ is the in-plane strain-displacements relative to q_I (the in-plane nodal displacements);
$\underset{\sim}{B}_0$ is the transverse strain-displacements relative to q_0 (the natural transverse nodal displacements).

The in-plane displacements q_I are

$$q_I = \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} \tag{38}$$

and the $\underset{\sim}{B}_I$ becomes

$$\underset{\sim}{B}_I = \frac{1}{J^*}\begin{bmatrix} \underline{g}_1^T\, \underset{\sim}{H}_n^T & \underline{o}^T \\ \underline{o}^T & \underline{g}_2^T\, \underset{\sim}{H}_n^T \\ \underline{g}_2^T\, \underset{\sim}{H}_n^T & \underline{g}_1^T\, \underset{\sim}{H}_n^T \end{bmatrix} \tag{39}$$

where

$$\tilde{H}_n^T = \underline{\partial}_n \underline{H}^T . \qquad (40)$$

The out of plane displacements $\underline{g}_0$ are

$$\underline{g}_0 = \begin{Bmatrix} g_{01} \\ g_{02} \\ g_{03} \\ g_{04} \end{Bmatrix} \qquad (41)$$

where the sub-elements of the partition are defined by equation (30) and $\tilde{B}_0$ becomes

$$\tilde{B}_0 = \tilde{I}_3 \; [\tilde{E} \; \tilde{\bar{H}}_n^T + \tilde{F} \; \tilde{\bar{H}}_{nn}^T] \qquad (42)$$

where

$$\tilde{I}_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad (43)$$

$$\tilde{\bar{H}}_n^T = \underline{\partial}_n \underline{\bar{H}}^T \qquad (44)$$

and

$$\underline{\bar{H}}_{nn}^T = \underline{\partial}_n^2 \; \underline{\bar{H}}^T . \qquad (45)$$

Note: In equation (37) the z variable, which is the plate's normal coordinate, is maintained distinctly since it is independent of the in-plane variables. Later, when the strain energy is formed, the z variable will integrate through the thickness and merge into material property matrices.

4. PLANE STRESS COMPONENTS

The stress components for a thin plate can be written in vector form as

$$\underset{-}{\sigma} = \tilde{C} \quad (\underset{-}{\varepsilon} - \underset{-}{\varepsilon}_T) \tag{46}$$

where $\tilde{C}$ is the material matrix described in the plate local axes and is expressed as

$$\tilde{C} = R_{\varepsilon}^T \; C \; R_{\varepsilon} \tag{47}$$

with C being the material matrix in the principal material directions of the fibers and R_{ε}^T being the strain transformation matrix from element local coordinates to principal fiber directions. The elements of equation (47) are found in reference 2.

The elements of the thermal strain involving the thermal coefficients are defined as

$$\tilde{\alpha} = R_{\varepsilon}^T \; \alpha \; . \tag{48}$$

Once the material matrix is defined, the elements of the material elasticity matrix can be defined as in the following section.

5. MATERIAL ELASTICITY MATRICES

The material coefficients are defined with the use of equation (3) in a slightly different form:

$$\int_V \underset{-}{\sigma}^T \underset{-}{\varepsilon} \, dV = \int_V \underset{-}{\varepsilon}^T \tilde{C} \, \varepsilon \, dV - \int_V \underset{-}{\varepsilon}^T \tilde{C} \, \underset{-}{\varepsilon}_T \, dV \tag{49}$$

Since the local z dimension is small compared to the x and y plate dimensions, it is convenient to define the stress resultants and moment resultants as

$$\underset{-}{N} = \int_t \underset{-}{\sigma} \, d\, z \tag{50}$$

and

$$\underset{\sim}{M} = \int_t \underline{\sigma}\, z\, d z \,. \tag{51}$$

Then, a new stress-strain matrix can be defined as

$$\bar{\underline{\sigma}} = \begin{Bmatrix} \underline{N} \\ \underline{M} \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B}^T & \underset{\sim}{D} \end{bmatrix} \begin{Bmatrix} \underline{\varepsilon}_o \\ \underline{\kappa} \end{Bmatrix} \tag{52}$$

where

$$\underset{\sim}{A} = \int_t \underset{\sim}{\tilde{C}}\, dz \tag{53}$$

$$\underset{\sim}{B} = \int_t \underset{\sim}{\tilde{C}}\, zdz \tag{54}$$

$$\underset{\sim}{D} = \int_t \underset{\sim}{\tilde{C}}\, z^2\, dz \,. \tag{55}$$

Letting

$$\underset{\sim}{E}_m = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B}^T & \underset{\sim}{D} \end{bmatrix} \tag{56}$$

$$\underset{\sim}{E}_T = \begin{bmatrix} \underline{A}_T & \underline{B}_T \\ \underline{B}_T & \underline{D}_T \end{bmatrix} \tag{57}$$

and

$$\bar{\underline{\varepsilon}} = \begin{Bmatrix} \underline{\varepsilon}_o \\ \underline{\kappa} \end{Bmatrix} \tag{58}$$

equation (49) can be written as

$$\int_V \underline{\sigma}^T\, \underline{\varepsilon}\, dV = \int_A \bar{\underline{\varepsilon}}^T\, \underset{\sim}{E}_m\, \bar{\underline{\varepsilon}}\, dA - \int_A \bar{\underline{\varepsilon}}^T\, \underset{\sim}{E}_T \begin{Bmatrix} T_o \\ T_g \end{Bmatrix} dA \tag{59}$$

where

$$\underset{-}{A}_T = \int_t \underset{\sim}{\tilde{C}}\, \underset{-}{\tilde{\alpha}}\, dz \qquad (60)$$

$$\underset{-}{B}_T = \int_t \underset{\sim}{\tilde{C}}\, \underset{-}{\tilde{\alpha}}\, z dz \qquad (61)$$

$$\underset{-}{D}_T = \int_t \underset{\sim}{\tilde{C}}\, \underset{-}{\tilde{\alpha}}\, z^2\, dz\ . \qquad (62)$$

The material $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$ matrices and the thermal load coefficients $\underset{-}{A}_T$, $\underset{-}{B}_T$ and $\underset{-}{D}_T$ can be related to laminar material by position t in the material build-up as

$$\underset{\sim}{A} = \sum_{i=1}^{L} \underset{\sim}{\tilde{C}}_i\, (t_i - t_{i-1}) \qquad (63)$$

$$\underset{\sim}{B} = 1/2 \sum_{i=1}^{L} \underset{\sim}{\tilde{C}}_i\, (t_i^2 - t_{i-1}^2) \qquad (64)$$

$$\underset{\sim}{D} = 1/3 \sum_{i=1}^{L} \underset{\sim}{\tilde{C}}_i\, (t_i^3 - t_{i-1}^3) \qquad (65)$$

$$\underset{-}{A}_T = \sum_{i=1}^{L} \underset{\sim}{\tilde{C}}_i\, \underset{-}{\tilde{\alpha}}_i\, (t_i - t_{i-1}) \qquad (66)$$

$$\underset{-}{B}_T = 1/2 \sum_{i=1}^{L} \underset{\sim}{\tilde{C}}_i\, \underset{-}{\tilde{\alpha}}_i\, (t_i^2 - t_{i-1}^2) \qquad (67)$$

$$\underset{-}{D}_T = 1/3 \sum_{i=1}^{L} \underset{\sim}{\tilde{C}}_i\, \underset{-}{\tilde{\alpha}}_i\, (t_i^3 - t_{i-1}^3) \qquad (68)$$

where the subscript "i" implies coefficient evaluation at laminae level "i" and L is the total number of fiber lamina levels.

6. COORDINATE TRANSFORMATIONS

The element information is initially determined in the natural coordinates of the plate since it is quite easy to express all loading and stiffness information in that reference. Ultimately the information must be transformed to local coordinates (x,y,z) and also to global coordinates (X,Y,Z). The following sections describe the transformations.

a. Natural to Local Transformation

The natural coordinate variables per node are defined as

$$\underset{\sim}{q}_{ni} = \begin{Bmatrix} u \\ v \\ w \\ w,_r \\ w,_s \\ \theta_z \end{Bmatrix}_i \tag{69}$$

where θ_z is the rotational degree of freedom normal to plate at node "i".

The transformation matrix required becomes

$$\underset{\sim}{q}_{ni} = t_{n\ell i} \; \underset{\sim}{q}_{\ell i} \tag{70}$$

where

$$\underset{\sim}{t}_{n\ell i} = \begin{bmatrix} I_{3\times 3} & \underset{\sim}{0} & \underline{0} \\ \underset{\sim}{0} & Q_{2\times 2} & \underline{0} \\ \underline{0}^T & \underline{0}^T & 1 \end{bmatrix}_i \tag{71}$$

$$\underset{\sim}{q}_{\ell i}^T = \lfloor u \; v \; w \; \theta_x \; \theta_y \; \theta_z \rfloor_i$$

with

$$\begin{Bmatrix} w,_r \\ w,_s \end{Bmatrix}_i = \underset{\sim}{Q}_i \begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix}_i \; . \tag{72}$$

Noting that the rotation degrees of freedom are defined as

$$\left.\begin{aligned} \theta_x &= w,_y \\ \theta_y &= -w,_x \end{aligned}\right\} \tag{73}$$

the $\underset{\sim}{Q}$ matrix becomes

$$\underset{\sim}{Q}_i = \begin{bmatrix} y,_r & -x,_r \\ y,_s & -x,_s \end{bmatrix}_i \tag{74}$$

The complete natural to local transformation therefore becomes

$$g_n = \begin{bmatrix} \underset{\sim}{t}_{n\ell 1} & & & \\ & \underset{\sim}{t}_{n\ell 2} & & \\ & & \underset{\sim}{t}_{n\ell 3} & \\ & & & \underset{\sim}{t}_{n\ell 4} \end{bmatrix} g_\ell \ . \tag{75}$$

b. Local to Global Transformations

The local to global transformation quantities are somewhat more difficult to obtain since the transformation involves the local coordinates of a quadrilateral element. Obviously only 3 points define a plane; the fourth point of the quadrilateral is unnecessary. The fourth point, however, may not lie in the same plane as the other three points. Therefore local transformations by node are determined and are averaged to obtain a general transformation used in determining the element coordinates and in transforming element matrices where applicable.

Defining the nodes of Figure 1 as i, j, k and l and allowing this sequence to permute, the element normal coordinate at node "i" which also permutes, is

$$\partial_{zi} = \frac{\underset{\sim}{V}_{ji} \times \underset{\sim}{V}_{il}}{|\underset{\sim}{V}_{ji} \times \underset{\sim}{V}_{il}|} \tag{76}$$

where the "X" symbols denote a cross product of two vectors, and $\underset{\sim}{V}_{ji}$ implies

$$\underline{V}_{ji} = \begin{Bmatrix} X_j - X_i \\ Y_j - Y_i \\ Z_j - Z_i \end{Bmatrix} \tag{77}$$

with X, Y and Z being global coordinates.

A global material reference $\underline{V}_m$ is defined as a global vector defined as input which locates the general local "x" of all elements referred to that vector. All material properties are defined relative to this x coordinate which becomes the element's local x coordinate. The element's local y coordinate at each node can be calculated as

$$\hat{e}_{yi} = \frac{\hat{e}_{zi} \times \underline{V}_m}{|\hat{e}_{zi} \times \underline{V}_m|} . \tag{78}$$

Finally, the local x coordinate at each node is determined as

$$\hat{e}_{xi} = \frac{\hat{e}_{yi} \times \hat{e}_{zi}}{|\hat{e}_{yi} \times \hat{e}_{zi}|} . \tag{79}$$

The local to global transformation at node "i" becomes

$$\underset{\sim}{t}_i = \begin{bmatrix} \hat{e}_x^T \\ \hat{e}_y^T \\ \hat{e}_z^T \end{bmatrix} \tag{80}$$

where the direction cosines of each coordinate are placed in row order in the transformation matrix.

The average transformation used to determine the local coordinates is obtained by first averaging the nodal normals as

$$\bar{\hat{e}}_z = \frac{1}{4} \sum_{i=1}^{4} \hat{e}_{zi} \tag{81}$$

and substituting into equations (78,79 and 80) to produce an average $\bar{t}_{\ell g}$.

The global degrees of freedom can be defined by node as

$$q_{gi} = \begin{Bmatrix} U \\ V \\ W \\ \theta_X \\ \theta_Y \\ \theta_Z \end{Bmatrix}_i \tag{82}$$

where U, V and W correspond to global displacements relative to X, Y and Z respectively and θ's corresponds to global rotations about X, Y and Z respectively.

The complete transformation becomes

$$\underset{\sim}{q}_\ell = \begin{bmatrix} \underset{\sim}{t}_{\ell g1} & & & \\ & \underset{\sim}{t}_{\ell g2} & & \\ & & \underset{\sim}{t}_{\ell g3} & \\ & & & \underset{\sim}{t}_{\ell g4} \end{bmatrix} \underset{\sim}{q}_g \tag{83}$$

where

$$\underset{\sim}{t}_{\ell gi} = \begin{bmatrix} \underset{\sim}{t} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{t} \end{bmatrix}_i \tag{84}$$

and $\underset{\sim}{q}_\ell$ and $\underset{\sim}{q}_g$ are the complete list of degrees of freedom per element.

If the quadrilateral element is perfectly flat in its space, then the $\underset{\sim}{t}_{\ell gi}$ becomes exactly $\underset{\sim}{\bar{t}}_{\ell g}$. If it is not, then the element space appears as a curved space. This effect should allow the element to behave as a shallow shell.

c. Natural to Global Transformation

The element stiffness will be transformed from natural to global coordinates directly. Therefore that transformation becomes

$$\underset{\sim}{g}_n = \underset{\sim}{T}_{ng}\, \underset{\sim}{g}_g \tag{85}$$

where

$$\underset{\sim}{T}_{ng} = \underset{\sim}{T}_{n\ell}\, \underset{\sim}{T}_{\ell g} \; . \tag{86}$$

Since both original transformation matrices are partitioned diagonally, the nodal transformation matrix is

$$\underset{\sim}{t}_{ngi} = \underset{\sim}{t}_{n\ell i}\, \underset{\sim}{t}_{\ell gi} \tag{87}$$

which contains many off-diagonal zeroes. Therefore it is convenient to define transformations by displacement and rotation degrees of freedom. That is, the displacement transformation at node i is

$$\underset{\sim}{t}_{Di} = \underset{\sim}{t}_i \tag{88}$$

and the rotation transformation at node i is

$$\underset{\sim}{t}_{Ri} = \begin{bmatrix} \underset{\sim}{Q}_i & \underset{\sim}{0} \\ \underset{\sim}{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{t}_i \end{bmatrix} . \tag{89}$$

This modification will save transformation operations later on.

7. ELEMENT STIFFNESS MATRIX

The element stiffness is easily defined in the natural coordinates of the plate, given the strain displacement functions. Once this stiffness is determined, it can be augmented with a scaffolding or artificial torsional stiffness for the plate's normal degrees of freedom. Finally, this stiffness can be transformed to local and then to global coordinates for assembly into a master stiffness matrix. The following details the above.

a. Plate Element Stiffness in Natural Coordinates

Defining a new set of degrees of freedom as

$$\bar{q}_n = \begin{Bmatrix} q_I \\ q_0 \end{Bmatrix}, \tag{90}$$

the strain components, as defined by equation (58) become

$$\bar{\varepsilon} = \underset{\sim}{B}_n \; \bar{q}_n \tag{91}$$

where

$$\underset{\sim}{B}_n = \begin{bmatrix} \underset{\sim}{B}_I & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_0 \end{bmatrix} \tag{92}$$

The second integral of equation (59) can be used to define the element stiffness in local coordinates as

$$\int_A \bar{\varepsilon}^T \; \underset{\sim}{E}_m \; \bar{\varepsilon} \; dA = \bar{q}_n^T \; \bar{\underset{\sim}{K}}_n \; \bar{q}_n \tag{93}$$

where

$$\bar{\underset{\sim}{K}}_n = \int_A \underset{\sim}{B}_n^T \; \underset{\sim}{E}_m \; \underset{\sim}{B}_n \; dA \; . \tag{94}$$

For convenience of computation, the material matrix $\underset{\sim}{E}_m$ in equation (94) is Cholesky factored as

$$\underset{\sim}{E}_m = \underset{\sim}{U}^T \; \underset{\sim}{U} \tag{95}$$

where $\underset{\sim}{U}$ is a upper tri-diagonal factoring matrix.

This allows equation (94) to be written in a more efficient form as

$$\bar{\underset{\sim}{K}}_n = \int_A (\underset{\sim}{U} \; \underset{\sim}{B}_n)^T \; (\underset{\sim}{U} \; \underset{\sim}{B}_n) \; dA \tag{96}$$

which allows the triple matrix product to be replaced by a simpler transpose symmetric product. This process is especially

efficient since the Cholesky factoring is performed (at most) once per element. Also, numerical integration is to be performed. Savings will occur at each integration point after the first.

b. Artificial Torsional Stiffness

The flat plate theory does not have any mechanism to directly include twisting of the plate normal to the plate surface. Therefore, if two coplanar elements are assembled at a common node, a singular stiffness exists. To avoid this, an artificial or scaffolding stiffness is added to the normal rotational degree of freedom θ_z. There is no change in the system equilibrium. For convenience, this is performed at all nodes of the element since it would be difficult to determine coplanar effects in general. This does change the overall element equilibrium. If the amount of artificial stiffness is kept small and the local rotational stiffness effects are in equilibrium, then the error can be minimized. Defining a vector of normal rotations at the nodes as

$$\underset{\sim}{\theta}_z = \begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \\ \theta_{z3} \\ \theta_{z4} \end{Bmatrix} , \qquad (97)$$

the artificial torsional stiffness matrix relative to $\underset{\sim}{\theta}_z$ becomes

$$\underset{\sim}{K}_{\theta z} = fC \begin{bmatrix} 3 & -1 & -1 & -1 \\ & 3 & -1 & -1 \\ & & 3 & -1 \\ \text{sym} & & & 3 \end{bmatrix} \qquad (98)$$

where f is an input scaling factor which can vary as

$$0 < f \leq 1 \tag{99}$$

and can be set in 1.E-8 increments, C is an artificial coefficient estimated from element bending stiffness coefficients and element area; i.e.,

$$C = \text{MIN } (D(1,1),\ D(2,2)) * \text{AREA} \tag{100}$$

with the D's defined in equation (55).

c. Natural Stiffness Matrix

The degrees of freedom $\bar{q}_n$ and θ_z defined by equations (90) and (97) can be merged to degrees of freedom q_n described in equation (69). This requires the re-ordering of stiffness coefficients of equations (96) and (98) to produce a natural stiffness matrix:

$$\underset{\sim}{K}_n = \begin{matrix}\text{MERGING}\\ \text{REORDERING}\end{matrix} \left[\bar{\underset{\sim}{K}}_n : \underset{\sim}{K}_{\theta z} \right] \tag{101}$$

relative to q_n.

d. Global Stiffness Matrix

The global stiffness matrix $\underset{\sim}{K}_g$ is formed by transforming $\underset{\sim}{K}_n$ from natural coordinates to global using equation (85). The transformation is formed using equation (1) realizing that the strain energy U_p is invariant relative to any coordinate reference. Therefore,

$$U_p = \frac{1}{2}\ q_n^T\ \underset{\sim}{K}_n\ q_n = \frac{1}{2}\ q_g^T\ \underset{\sim}{K}_g\ q_g\ . \tag{102}$$

Using equation (85) produces

$$\underset{\sim}{K}_g = \underset{\sim}{T}_{\ell g}^T\ \underset{\sim}{K}_n\ \underset{\sim}{T}_{\ell g}\ . \tag{103}$$

The triple matrix product implied in equation (103) is quite inefficient, especially since $\underset{\sim}{T}_{\ell g}$ is highly diagonal. Efficiency

can however be effected by partitioning $\underset{\sim}{K}_n$ and $\underset{\sim}{K}_g$ into 3 X 3 sub-matrices labelled $\underset{\sim}{K}_{ij}^{n}$ and $\underset{\sim}{K}_{ij}^{g}$ where i and j range from 1 to 8. Then

$$\underset{\sim}{K}_{ij}^{g} = \underset{\sim}{t}_i^{T} \ \underset{\sim}{K}_{ij}^{n} \ \underset{\sim}{t}_j \qquad (104)$$

where $\underset{\sim}{t}_i$ matrix relates to equation (88) when i equal 1, 3, 5, 7 and relates to equation (89) when i equals 2, 4, 6, 8.

Additional efficiency is obtained when the triple matrix product is performed such that the right portion matrix multiply is first formed and positioned back into $\underset{\sim}{K}_{ij}^{n}$. Then, the left multiply is formed with the resulting product and placed into $\underset{\sim}{K}_{ij}^{g}$.

6. NUMERICAL INTEGRATION OF AREA FUNCTIONS

The elements in the matrix of equations (94) and (96) are very difficult to integrate exactly, therefore approximate numerical integration can be performed with sufficient accuracy for justification. Gauss-Legendre Numerical Quadrature has been selected to perform the integration of the stiffness coefficients as well as other area functions. Equation (94) can be rewritten and transformed relative to variables and limits of integrations as

$$\underset{\sim}{K}_n = \int_{-1}^{1} \int_{-1}^{1} \underset{\sim}{B}_n^{T} (r,s) \ \underset{\sim}{E}_m \underset{\sim}{B}_n (r,s) \ J^* \, dr ds . \qquad (105)$$

An element of this matrix can be written as

$$k_{ij} = \int_{-1}^{1} \int_{-1}^{1} f_{ij} (r,s) \, dr ds \qquad (106)$$

where

$$f_{ij}(r,s) = \sum_{k} \underset{\sim}{B}_n^{T} (k,i) \sum_{\ell} \underset{\sim}{E}_m(k,\ell) \ \underset{\sim}{B}_n(\ell,j) \ J^* . \qquad (107)$$

The stiffness coefficient can then be approximated as

$$k_{ij} = \sum_{k}^{n_1} \sum_{\ell}^{n_2} f_{ij} (r_k, S_\ell) W_k W_\ell \tag{108}$$

where n_1, n_2 are the number of Legrendre root evaluation points in the r,s directions respectively.
r_k and s_ℓ are the roots of the Legrendre polynomial,
W_k and W_ℓ are the appropriate Gauss weighting factors.

For

$$n = 2, \qquad r_k = s_k = -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \qquad W_k = 1, 1 \tag{109}$$

For

$$n = 3, \qquad r_k = s_k = -\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}} \qquad W_k = \frac{5}{9}, \frac{8}{9}, \frac{5}{9} \tag{110}$$

9. ELEMENT MASS MATRIX

A consistent mass matrix relative to the in-plane variables in one coordinate can be written as

$$\underset{\sim}{M}^c = \int_A \rho\, t\, \underline{H}\, \underline{H}^T\, dA \tag{111}$$

and a corresponding lumped mass matrix can be formed by summing the rows of the consistent mass matrix as

$$\underline{M}^\ell = \rho\, t \int_A \underline{H}^T\, dA \tag{112}$$

assuming t and ρ constant and realizing

$$\sum_{j=1}^{4} H_j = 1.$$

The components of $\underline{M}^{\ell}$ are applied to the translatory degrees of freedom in all global directions, per node. A rotary inertia effect can be included by an approximation;

$$m_{Ri} = m_i \frac{t^2}{12} . \tag{113}$$

Finally, the global lumped mass vector can be formed:

$$\underline{M}_g^{\ell} = \begin{Bmatrix} m_1 & \underline{1} \\ m_{R1} & \underline{1} \\ m_2 & \underline{1} \\ m_{R2} & \underline{1} \\ \vdots & \end{Bmatrix} \tag{114}$$

where $\underline{1}$ is a 3 X 1 unit vector.

10. ELEMENT LOAD VECTORS

The element load vectors are established from element properties such as material constants, temperatures, pressure, mass, area and acceleration constants. The following sections describe the load vectors developed.

a. Thermal Load Vector

The third integral of equation (59) can be used to define the element thermal load vector using equation (91):

$$\int \underline{\bar{\varepsilon}}^T \; \underset{\sim}{E}_T \begin{Bmatrix} t_o \\ t_g \end{Bmatrix} dA = \underline{q}_n^T \int_A \underset{\sim}{B}_n^T \; dA \; \underset{\sim}{E}_T \begin{Bmatrix} t_o \\ t_g \end{Bmatrix} . \tag{115}$$

Therefore,

$$\underline{F}_n^T = \left[\int_A \underset{\sim}{B}_n^T \; dA \right] \; \underline{\bar{\sigma}}_T \tag{116}$$

is the thermal vector relative to the natural coordinates and $\underline{\bar{\sigma}}_T$ is the thermal stress vector formed from strains:

$$\tilde{\sigma}_T = \tilde{E}_T \begin{Bmatrix} t_o \\ t_g \end{Bmatrix} . \tag{117}$$

The global thermal vector can be determined by applying the natural to global transformation,

$$\underline{F}_g^T = \tilde{T}_{ng}^T \; \underline{F}_n^T . \tag{118}$$

b. Pressure Load Vector

The pressure load, by node, is formed for the shape terms associated only with the geometry and is applied in the z (plate normal) direction. Therefore,

$$\underline{F}_z^p = \int_A p \; \underline{H} \; dA . \tag{119}$$

A pressure load vector can be formed relative to local coordinates $\underline{u}_{\ell i}$ as

$$\underline{F}_{\ell i}^p \begin{Bmatrix} 0 \\ 0 \\ f_{zi}^p \\ 0 \\ 0 \\ 0 \end{Bmatrix} . \tag{120}$$

The global pressure load can be formed as

$$\underline{F}_g^p = \tilde{T}_{\ell g}^T \; \underline{F}_\ell^p . \tag{121}$$

which transforms the normal traction into a global traction.

c. Constant Acceleration Load Vector

The acceleration vector can be computed from the mass vector defined in equation (114); i.e.,

$$\underline{F}_g^a = \tilde{a} \; \underline{M}_g^\ell \tag{122}$$

where

$$\underset{\sim}{a} = \begin{bmatrix} \underset{\sim}{a}^* & & & \\ & \underset{\sim}{a}^* & & \\ & & \underset{\sim}{a}^* & \\ & & & \underset{\sim}{a} \end{bmatrix} \qquad (123)$$

with

$$\underset{\sim}{a}^* = \begin{bmatrix} a_X & & & & & \\ & a_y & & & & \underset{\sim}{0} \\ & & a_Z & & & \\ & & & 0 & & \\ & \underset{\sim}{0} & & & 0 & \\ & & & & & 0 \end{bmatrix} \qquad (124)$$

where a_X, a_Y, a_Z are the acceleration coefficients in the X, Y, and Z coordinates respectively.

Equations (118), (121) and (122) imply multiplications of full matrices, the T transformation matrices containing many zeros. Actually, the multiplications of matrices are done [illegible] sub-element for efficiency, then placed in proper matrix positions.

11. STRESS RECOVERY MATRIX

Once the displacements of the element [illegible] the stresses can be determined [illegible] of the mid-plane surface. [illegible] and (117), the stress recovery [illegible]

$$[illegible] \qquad (125)$$

where the [illegible] matrix [illegible]

$$\underset{\sim}{S} = \underset{\sim}{E}_m \underset{\sim}{B}_n \underset{\sim}{T}_{ng} \ . \qquad (126)$$

The elements of $\underset{\sim}{S}$ and $\underline{\bar{\sigma}}_T$ are saved on a secondary storage device for recovery once the displacements are computed.

SECTION III

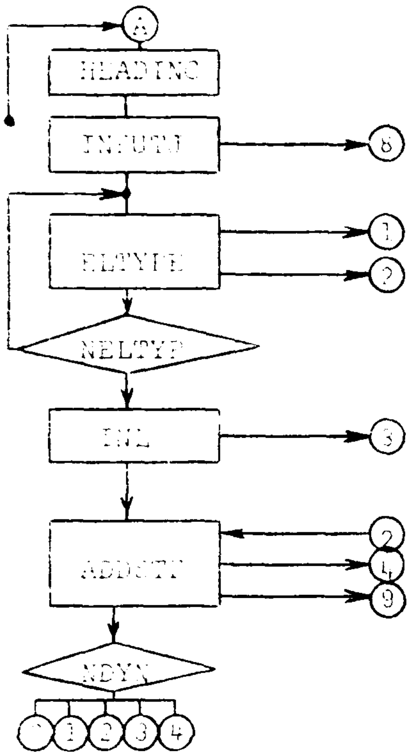
MODIFICATIONS TO SAP IV COMPUTER PROGRAM

The SAP IV computer program was modified to accept the new composite element. The following sections describe changes to the existing program as well as new routines of the composite plate element.

1. SAP IV STRUCTURE

The main changes to the SAP IV program occur in the element library control. They occur in the element generation portion of the program. Figures 2 and 3 depict the major routines used by SAP to perform a static analysis of its elements. The ELTYPE routine calls a new routine called PLATE, which is used to call OVERLAY (8). (stress recovery) Once OVERLAY (8) is called, a call is made to CPLATE, the main routine of the composite plate finite element. Figures 4, 5, 6, and 7 describe the flow of routines used by CPLATE.

FLOW TAPE UNITS DESCRIPTION



Input title and master control card

Input nodes
Write 8 ID, X, Y, Z, T

Call element type (new composite plate element called here)
WRITE 1 Element stress matrices for recovery
WRITE 2 Element stiffness matrices, mass and load vectors

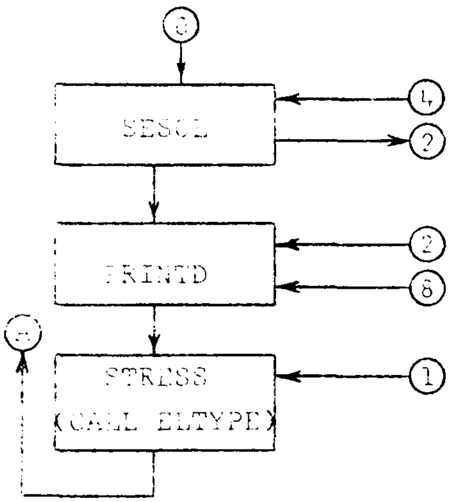
Input Concentrated loads/masses
WRITE 3 Concentrated loads/ masses for assembly

READ 2 Read stiffness for assembly
WRITE 4 Assembled Stiffness and Loads in blocks
WRITE 9 Assembled Mass Vector

Call appropriate solution type
0 Static
1 Nodal Extraction
2 Force Response Analysis
3 Response Spectrum Analysis
4 Direct Integration

FIGURE 2 BASIC PROGRAM FLOW

FLOW   TAPE UNITS   DESCRIPTION



Static solution option

Solution of displacement equation
READ 4   Read stiffness and loads for decomposition
WRITE 2   Solve for displacements

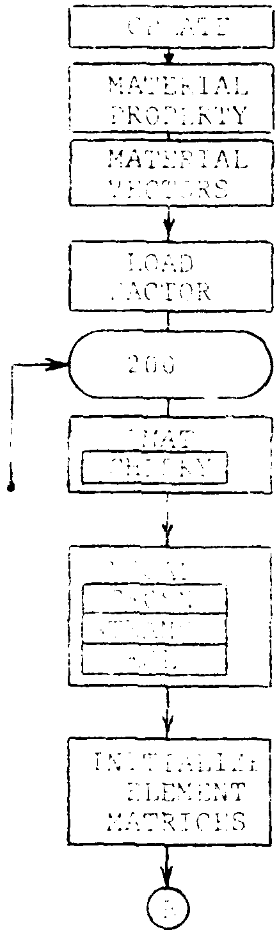
READ 8   Read ID identification matrix
READ 2   Recover displacements
Print displacements

READ 1   Element stress matrices
Stress recovery and printing

FIGURE · STATIC SOLUTION AND RECOVERY STAGE

TAPE UNITS

DESCRIPTION



Composite plate routine

Input material properties in tables

Input global material vectors

Input load factors TLO as a function of thermal, pressure and constant acceleration

Input element definition

Form ABD matrices and calculate $\bar{\sigma}_T$ thermal stress vector. Modify $\bar{y}^T \underset{\sim}{E}_m = \underset{\sim}{U}^T \underset{\sim}{U}$

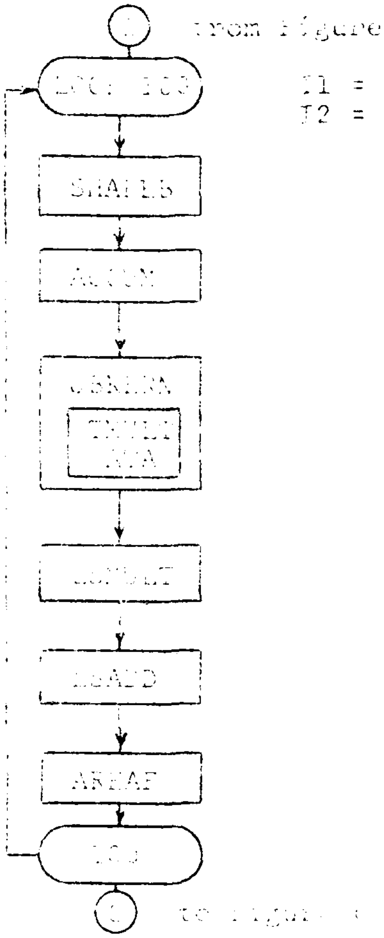
Determine local to global transformation and local element coordinates.
CROSN - forms vector cross product
STRANS - special transpose
KGL - used to transform global to
local coordinates

Set element K_n, B_n, A, M_g^ℓ matrices to zero.

to Figure 5

FIGURE 4 FLOW DIAGRAM FOR CPLATE

DATA UNITS | DESCRIPTION



$I1 = 1,NG$
$I2 = 1,NG$

Loop on NG (Gauss points) in each direction

Determine natural shape function $\underset{\sim}{B}_n$ (I1, I2)

Accumulate $\underset{\sim}{B}_n$ over all Gauss points

TMULT - forms special multiply of $U\ B_n$

ATA - form special symmetric multiply $(U\ \underset{\sim}{B}_n)^T(U\ \underset{\sim}{B}_n)$

Forms triple matrix product and stores result in $\underset{\sim}{K}_n$

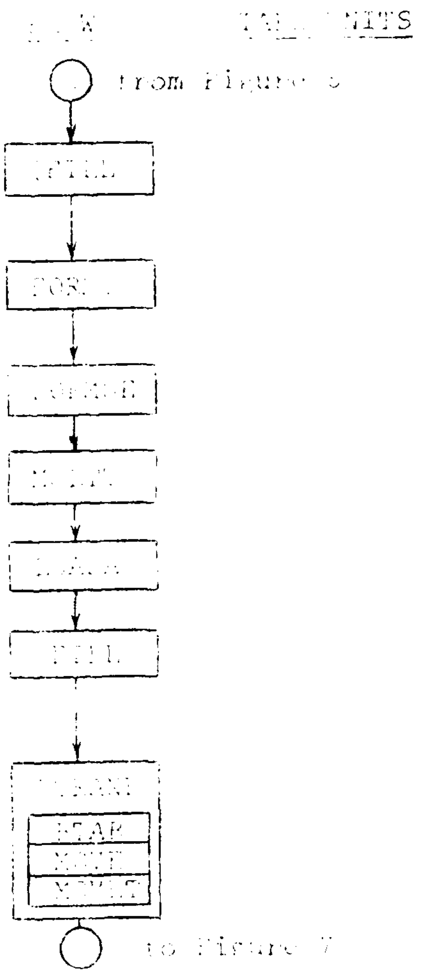
Lower symmetric scaler multiply - forms $\underset{\sim}{K}_n$ (I1,I2) = W(I1)*W(I2)*$\underset{\sim}{K}_n$

Lower symmetric add - ads integration results to accumulator for local element stiffness.

Determines area distribution vector $\underline{A}$ by node.

FIGURE 5 FLOW DIAGRAM FOR CPLATE (CON'T)

DESCRIPTION



Fill the $\underset{\sim}{Q}$ matrix with natural to local coordinate transformation quantities.

Form special TT transformation from natural to global coordinates.

Forms $\underset{\sim}{B}_{g} = \underset{\sim}{r}_{n} \, \underset{\sim}{T}_{ng}$ for the stress matrix.

Forms matrix multiply to produce thermal load vector.

Calculate pressure, acceleration and mass vectors.

Prepare $\underset{\sim}{K}_{g}$ stiffness matrix with $\underset{\sim}{K}_{n}$ and $\underset{\sim}{K}_{0z}$

Transforms $\underset{\sim}{K}_{n}$ to $\underset{\sim}{K}_{g}$

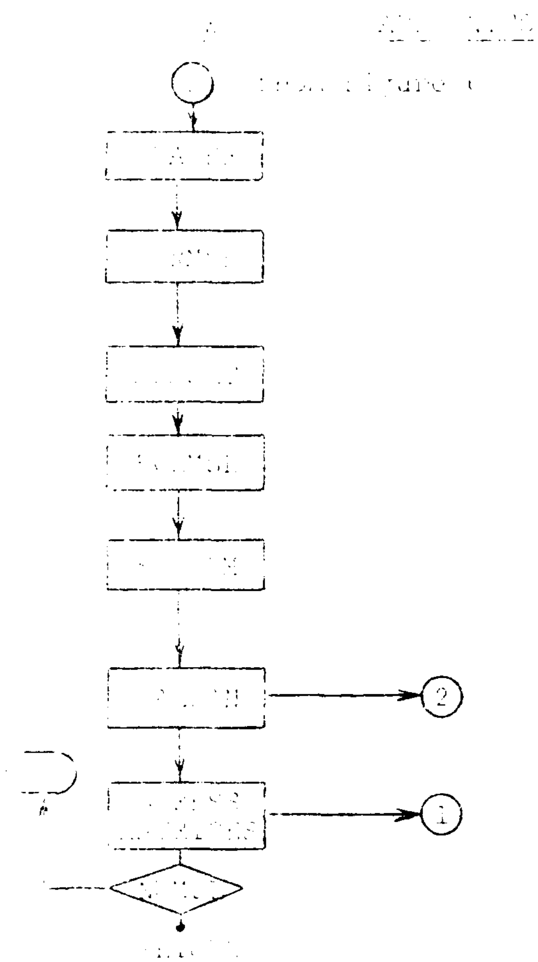
BTAB - performs special triple matrix product on sub-partition stiffness

MOVE - move sub-matrix into $\underset{\sim}{K}_{g}$

MOVET - move sub-matrix into $\underset{\sim}{K}_{g}$ (symmetry transpose)

FIGURE 6 FLOW DIAGRAM FOR CPLATE (CON'T)

DESCRIPTION



Form B_n at stress evaluation point

Form $\ldots$, the strain displacement function for stress recovery.

Special matrix multiply -
$B_n = E_m B_n$

Form element stress matrix relative to global displacements.

Set location vector of equation numbers for element global connectivity.

Calculate band width of equation
WRITE 2 stiffness, load vectors and mass vectors for assembly

WRITE 1 stress matrix for stress recovery

Transfer control to main program

FIGURE 7 FLOW DIAGRAM FOR CPLAST (CON'T)

[illegible] ROUTINES

[illegible] contains a description of the routines used [illegible] composite plate overlay.

[illegible] - a subroutine called by CPLATE to fill the $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$ of equations 53, 54 and 55 material matrices from the input material [illegible] arrays. Equations 60, 61, and 62 are used to [illegible] the thermal load vectors and thermal [illegible] of equation 117.

[illegible] - a subroutine called by EMAP to factor the material matrix $\tilde{E}_m$ into an upper triangular matrix as described in equation 75.

[illegible] - a subroutine called by CPLATE to determine the local to global transformation matrix as described by equations 76 through [illegible]. The routine checks for proper area definitions.

[illegible] - a subroutine called by LOCAL to perform vector cross products [illegible] in equations 76, 78 and 79. The resulting vector components are normalized to unit vectors.

[illegible] - a subroutine called by LOCAL to perform an in-place square matrix transpose of the local to global transformation of equation 80.

[illegible] - a subroutine called by LOCAL to transform global element [illegible] to local element coordinates. Transformation matrix described by equation 80.

[illegible] - a subroutine called by CPLATE to set matrix array space to any value. Specifically, it is used to set matrix space to zero.

[illegible] - a subroutine called by CPLATE to form composite plate element strain displacement function $\tilde{B}_n$ as described by equation [illegible] and uses equations [illegible] through 45.

[illegible] - a subroutine called by CPLATE to accumulate the results of numerical quadrature of $\tilde{B}_n$ dA over all Gauss points of the area as used in equation 116.

[illegible] - a subroutine called by CPLATE to form a special form of the [illegible] product described in equation 94 and [illegible]. This routine is designed for efficiency during the integration procedure.

UMULT - a subroutine called by UBKERN to perform a special matrix multiply where the leading matrix is an upper triangular matrix; described by equation 96.

ATA - a subroutine called by UBKERN to perform a special symmetric multiply as needed in equation 96.

LSMULT - a subroutine called by CPLATE to perform a lower symmetric scalar multiply of the natural stiffness components during the numerical integration, as in equation 108.

LSADD - a subroutine called by CPLATE to perform the lower symmetric addition of the stiffness matrix components during numerical integration, as described by equation 108.

AREA - a subroutine called by CPLATE to determine the distribution of area, by node, for the quadrilateral plate element. This area function is needed in equations 112 and 119.

QFILL - a subroutine called by CPLATE to form a transformation matrix from natural to local coordinates as described in equation 74.

FORMT - a subroutine called by CPLATE to form the natural to global coordinate transformation as shown in equation 87. Since the transformation matrix is diagonal, only the diagonal sub-matrices are stored.

FORMSN - a subroutine called by CPLATE to perform a natural to global transformation of the strain-displacement and stress-displacement matrices described in equations 116 and 120.

MULT - a subroutine called by CPLATE, FORMT and FORMSN to perform a general matrix multiplication of arbitrary matrices selected from or position to any sub-matrix position.

LOADS - a subroutine called by CPLATE to calculate the element pressure, constant acceleration and mass vectors described by equations 112, 119 and 122.

FILL - a subroutine called by CPLATE to prepare the elements of the element global stiffness matrix with elements from the natural stiffness and artificial torsional stiffness matrices. This process is described in equation 101.

PTRANSF - a subroutine called by CPLATE to transform, by node, the sub-matrices of the natural stiffness matrix into corresponding global stiffness matrices. The original natural stiffness matrix and the final global stiffness matrix occupy the same matrix space. This procedure is described by equation 103.

ATAB - a subroutine called by PTRANSF to perform a special triple matrix product used in stiffness transformations. This routine performs an efficient double matrix multiply with an over-write of the original sub-matrix as described by equation 104.

MOVE - a subroutine called by PTRANSF to move the elements of a sub-matrix of any rank and place them into new matrix positions.

MOVET - a subroutine called by PTRANSF to move elements similarly to MOVE except that, the receiving sub-matrix is the matrix transposed.

LASMUL - a subroutine called by CPLATE to perform a special matrix multiply such that the post multiplying matrix is over written as shown in equation 126.

SECTION IV

PROGRAM VERIFICATION

[illegible] to the SAP IV program produce a new version named [illegible] which basically includes an additional element, TYPE 8, (composite [illegible] element). The original SAP IV program, having many different elements and many different static and [illegible] analysis procedures, has been already verified; therefore only the new element was checked under various boundary and loading conditions.

The new element was compared with many simple degenerate [illegible]-to-[illegible] type problems and was found to produce excellent [illegible]. The same tests were performed for element TYPE 6 and [illegible] results were more favorable for the simple cases. [illegible] TYPE 8 results in larger displacements while TYPE 6 generally produces lower than exact solution values.

The following section describes a group of verification [illegible] plates, curved shells and a doubly curved blade. Most [illegible] solutions were obtained by classical plate and shell techniques with composite material properties. The procedure [illegible] using Fourier series to approximate solutions of various [illegible] boundary conditions. Once the A, [illegible], D matrices were determined and boundary conditions applied, an approximate solution was formed using a seven or eight-term expansion. Most [illegible] used four to eight elements per direction. Convergence was not specifically studied since a general study of the element was made in Reference 2.

[illegible] CONCENTRATED LOAD AT CENTER [illegible]
[illegible] SUPPORTED

[illegible]

= [illegible]" [illegible] = [illegible]" t = .2"

[illegible]

[illegible] .[illegible]7[illegible]
[illegible] x [illegible]
= .[illegible]

[illegible]

= [illegible] lb.

[illegible] conditions:

[illegible] = [illegible], $M_x = 0$

[illegible] = [illegible], $M_y = 0$

[illegible] at center, W_c

([illegible]) exact = .[illegible]16

([illegible]) [illegible] = .[illegible]644

([illegible]) exact = $\dfrac{(\,[illegible]\,)_{ex.}\, E\, h^3}{p\, a^4} \times 10^2 = .1267$

([illegible]) [illegible] = $\dfrac{(\,[illegible]\,)_{ap.}\, E\, h^3}{p\, a^4} \times 10^2 = .1262$

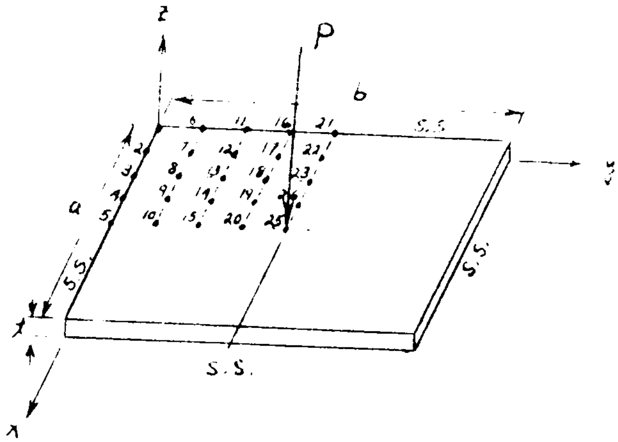


FIGURE 8

. . ISOTROPIC PLATE UNDER UNIFORMLY DISTRIBUTED LATERAL LOAD WITH ALL EDGES SIMPLY-SUPPORTED

(1) Size of plate:

$a = 10''$ $b = 10''$ $t = .2''$

(2) Properties of plate:

$\rho = .0275\ ^{lb}/in^3$
$E = 30 \times 10^6$ psi
$\nu = .3$

(3) Loading condition:

$p\ (x,y) = p_o$

(4) Boundary conditions:

$x = 0,\ a;\quad w = 0 \quad M_x = 0$
$y = 0,\ b;\quad w = 0 \quad M_y = 0$

(5) Deflection at center, W_c;

$(W_c)_{\text{exact}} = .18437 \times 10^{-2}\ po\ in^3/lb.$
$(W_c)_{\text{sap.}} = .13351 \times 10^{-2}\ po\ in^3/lb.$

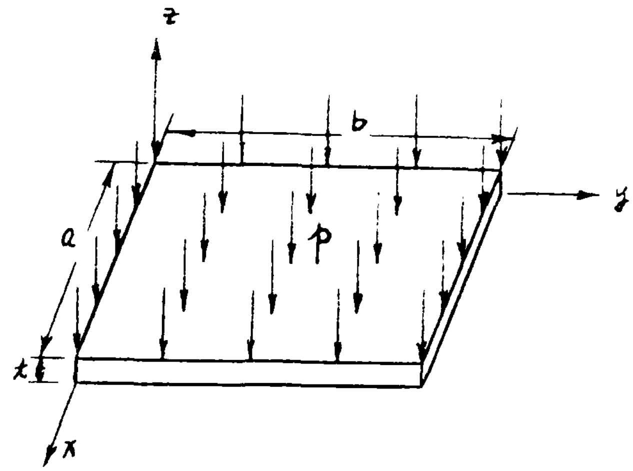


FIGURE 9

5. $[90/0_2/90]_T$ UNDER UNIFORMLY DISTRIBUTED LATERAL LOAD WITH ALL EDGES SIMPLY-SUPPORTED

(1) Size of laminated plate:

$a = 10'' \quad b = 10'' \quad t = 4h = .2''$

(2) Properties of plate:

$\rho = .0275 \ ^{lb}/in^3$

$$\underset{\sim}{A} = \begin{bmatrix} 3.3 & .2 & 0 \\ .2 & 3.3 & 0 \\ 0 & 0 & .2 \end{bmatrix} \times 10^6 \ ^{lb}/in$$

$$\underset{\sim}{B} = 0$$

$$\underset{\sim}{D} = \begin{bmatrix} 4.25 & .67 & 0 \\ .67 & 17.75 & 0 \\ 0 & 0 & .67 \end{bmatrix} \times 10^3 \ \text{lb-in}$$

(3) Loading condition:

$p(x,y) = p_o$

(4) Boundary conditions:

$x = 0, a; \quad w = 0 \quad M_x = 0$

$y = 0, b; \quad w = 0 \quad M_y = 0$

(5) Deflection at center, W_c;

$(W_c)_{\text{exact}} = 62.38 \times 10^{-4} \ p_o \ ^{in^3}/lb$

$(W_c)_{\text{sap.}} = 62.34 \times 10^{-4} \ p_o \ ^{in^3}/lb$

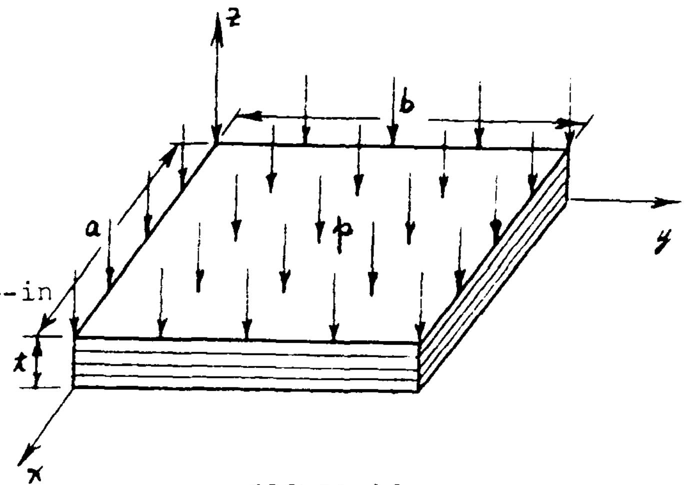


FIGURE 10

4. $[0_2/90_2]_t$ UNDER UNIFORMLY DISTRIBUTED LATERAL LOAD WITH ALL EDGES SIMPLY-SUPPORTED

(1) Size of laminated plate:

$a = 10''$ $b = 10''$ $t = 4h = .2''$

(2) Properties of plate:

$\rho = .0275\ ^{lb}/in^3$

$$\underset{\sim}{A} = \begin{bmatrix} 3.3 & .2 & 0 \\ .2 & 3.3 & 0 \\ 0 & 0 & .2 \end{bmatrix} \times 10^6\ ^{lb}/in$$

$$\underset{\sim}{B} = \begin{bmatrix} .135 & 0 & 0 \\ 0 & -.135 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^6\ lb$$

$$\underset{\sim}{D} = \begin{bmatrix} 11 & .67 & 0 \\ .67 & 11 & 0 \\ 0 & 0 & .67 \end{bmatrix} \times 10^3\ lb\text{-}in$$

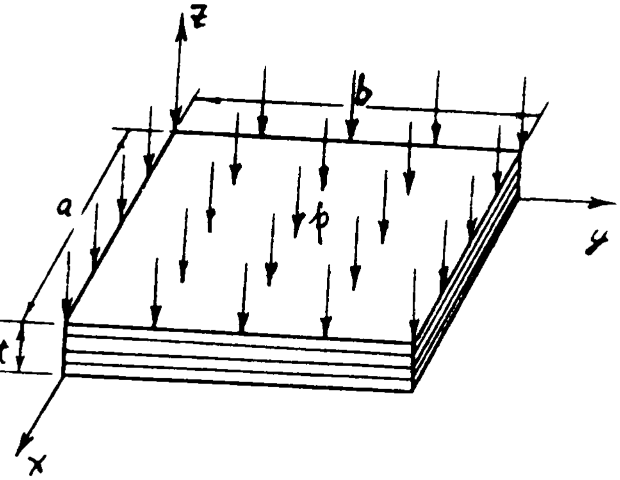


FIGURE 11

(3) Loading condition:

$p(x,y) = p_o$

(4) Boundary conditions:

$x = 0, a;$ $w = 0$ $M_x = 0$ $v = 0$ $N_x = 0$

$y = 0, b;$ $w = 0$ $M_y = 0$ $u = 0$ $N_y = 0$

(5) Deflection at center, W_c;

$(W_c)_{exact} = 114.4 \times 10^{-4}\ p_o\ in^3/lb.$

$(W_c)_{sap.} = 113.6 \times 10^{-4}\ p_o\ in^3/lb.$

5. $[0/90]_t$ UNDER IN-PLANE LOAD WITH TWO EDGES PERPENDICULAR TO THE DIRECTION OF LOAD FREE AND OTHER TWO EDGES SIMPLY-SUPPORTED

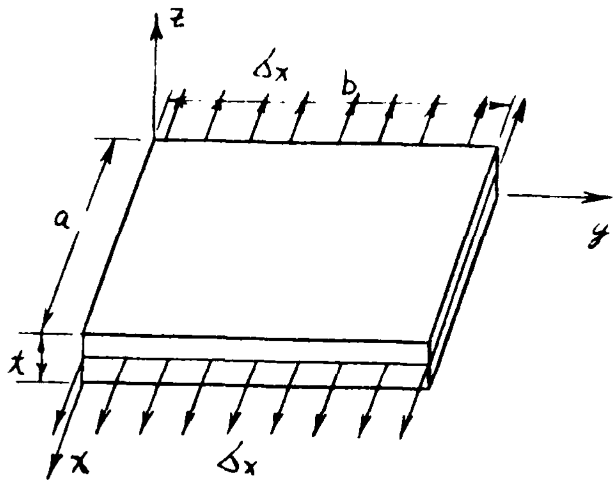


FIGURE 12

(1) Size of laminated plate:

a = 10" b = 10" t = 2h = .2"

(2) Properties of plate:

$\rho = .2075 \text{ lb/in}^3$

$$\tilde{A} = \begin{bmatrix} 1.65 & .1 & 0 \\ .1 & 1.65 & 0 \\ 0 & 0 & .1 \end{bmatrix} \times 10^6 \text{ lb/in}$$

$$\tilde{B} = \begin{bmatrix} 3.37 & 0 & 0 \\ 0 & -3.37 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4 \text{ lb}$$

$$\tilde{D} = \begin{bmatrix} 13.8 & .83 & 0 \\ .83 & 13.8 & 0 \\ 0 & 0 & .83 \end{bmatrix} \times 10^2 \text{ lb-in}$$

(3) Loading conditions:

$\sigma_x = 1$ psi

(4) Boundary conditions:

x = 0 ; u = 0

x = a ; u ≠ 0

(5) Curvature at x-direction, K_x

$(K_x)_{exact} = -.305\sigma_x$

$(K_x)_{sap.} = -.3057\sigma_x$

6. $[0/90]_T$ UNDER FREE VIBRATION WITH ALL EDGES SIMPLY-SUPPORTED

(1) Size of laminated plate:

$a = 10"$ $b = 10"$ $t = 2h = .2"$

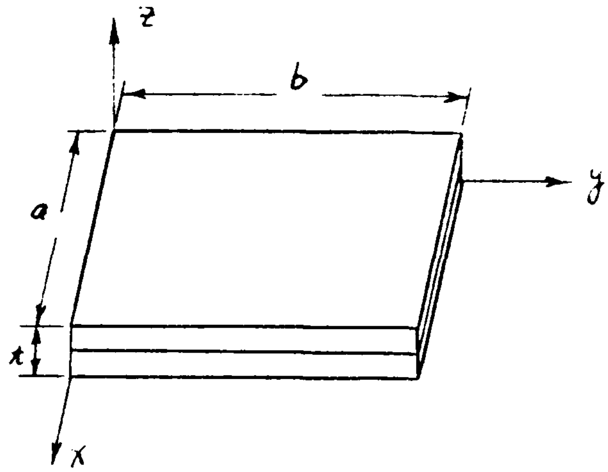


FIGURE 13

(2) Properties of plate:

$\rho = .0275 \text{ lb/in}^3$

$$\underset{\sim}{A} = \begin{bmatrix} 1.65 & .1 & 0 \\ .1 & 1.65 & 0 \\ 0 & 0 & .1 \end{bmatrix} \times 10^6 \text{ lb/in}$$

$$\underset{\sim}{B} = \begin{bmatrix} 3.37 & 0 & 0 \\ 0 & -3.37 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4 \text{ lb}$$

$$\underset{\sim}{D} = \begin{bmatrix} 13.8 & .83 & 0 \\ .83 & 13.8 & 0 \\ 0 & 0 & .83 \end{bmatrix} \times 10^2 \text{ lb-in}$$

(3) Loading condition:

free vibration

(4) Boundary conditions:

$x = 0, a;$ $\delta w = 0,$ $\delta M_x = 0,$ $\delta v = 0,$ $\delta N_x = 0$

$y = 0, b;$ $\delta w = 0,$ $\delta M_y = 0,$ $\delta u = 0,$ $\delta N_y = 0$

(5) Frequency:

$(f)_{exact} = 8.990$ Hz.

$(f)_{sap.} = 8.887$ Hz.

7. $[90/0_2/90]_T$ CURVED PLATE UNDER UNIFORM PRESSURE WITH ALL EDGES SIMPLY-SUPPORTED

(1) Size of laminated curved plate:

$a = 10"$ $b = 10.09"$ $t = 4h = .2"$
$s = 10"$ $R = 21.23"$

(2) Properties of plate:

$\rho = .0275 \text{ lb/in}^3$

$$\underset{\sim}{A} = \begin{bmatrix} 3.3 & .2 & 0 \\ .2 & 3.3 & 0 \\ 0 & 0 & .2 \end{bmatrix} \times 10^6 \text{ lb/in}$$

$$\underset{\sim}{B} = \underset{\sim}{0}$$

$$\underset{\sim}{D} = \begin{bmatrix} 4.25 & .67 & 0 \\ .67 & 17.75 & 0 \\ 0 & 0 & .67 \end{bmatrix} \times 10^3 \text{ lb-in}$$

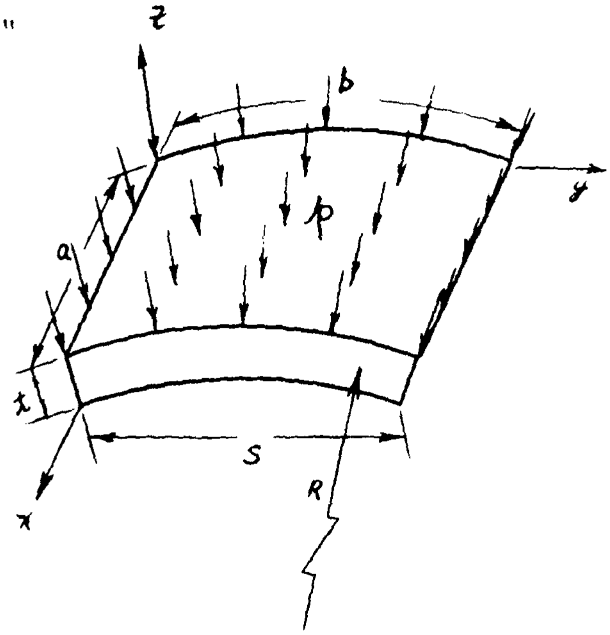


FIGURE 14

(3) Loading condition:

$p(x,y) = p_o$

(4) Boundary conditions:

$x = 0, a;$ $w = 0,$ $M_x = 0,$ $v = 0,$ $N_x = 0$
$y = 0, b;$ $w = 0,$ $M_y = 0,$ $u = 0,$ $N_y = 0$

(5) Deflection at center W_c;

$(W_c)_{\text{exact}} = 24.04 \times 10^{-4}\ p_o \text{ in}^3/\text{lb}.$
$(W_c)_{\text{sap.}} = 23.68 \times 10^{-4}\ p_o \text{ in}^3/\text{lb}.$

8. BI-METALLIC STRIP UNDER UNIFORM HEATING WITH ALL EDGES SIMPLY-SUPPORTED

(1) Size of the beam:

$a = 10"$ $b = 10"$ $t = .2"$

(2) Properties of the beam:

$\bar{\rho} = .0275\ \mathrm{lb/in}^3$

$E_{steel} = 30 \times 10^6$ psi

$E_{sl} = 10 \times 10^6$ psi

$\alpha_{steel} = 6.5 \times 10^{-6}\ {}^{\circ}F^{-1}$

$\alpha_{sl} = 10.5 \times 10^{-6}\ {}^{\circ}F^{-1}$

(3) Loading condition:

$\Delta T = 100^{\circ}F$

(4) Boundary conditions:

$x = 0;\ u = 0,\ M_x = 0$

$y = 0;\ v = 0,\ M_y = 0$

(5) Curvature at x-direction, K_x;

$(K_x)_{exact} = .01394$

$(K_x)_{sap.} = .01385$

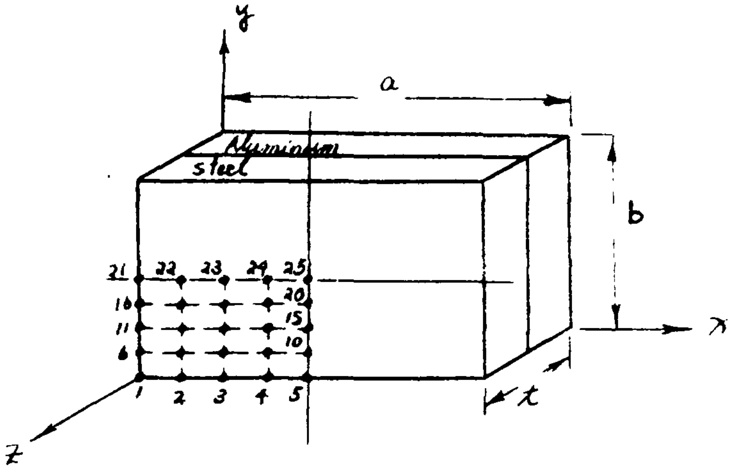


FIGURE 15

5. NATURAL FREQUENCY OF A TYPICAL BLADE CONFIGURATION

(1) Type of Blade: (Figure 16)

A typical blade used in aircraft engines, named J79 B/AL, was approximated using 64 quadrilateral plate elements in SAP4A (both TYPE 6 and TYPE 9). An equivalent NASTRAN model was also run along with an experimental test to determine fundamental frequency at zero frequence (results from AFSC - Wright-Patterson AFB).

(2) Properties of blade:

Material used corresponded to the input used in the NASTRAN run using anisotropic material properties.

Leading edge:

ρ = .000407 lb sec^2/in

Cxx = 26.9E8 lb/in^2 Cxy = 4.6E7 lb/in^2

Cyy = 20.3E8 lb/in^2 Gxy = 7.3E7 lb/in^2

Blade:

ρ = .000251 lb sec^2/in

Same material coefficients as above.

(3) Loading Condition:

Mass and Stiffness distributions for eigensolution.

(4) Boundary Conditions:

Base of blade completely fixed and rest of blade free.

(5) Natural Frequency of Blade: (Fiest FLEX)

(f)exp = 110 Hz.

(f)Nastran = 106.7 Hz.

(f)sap (T6) = 108.3 Hz.

(f)sap (T9) = 112.8 Hz.

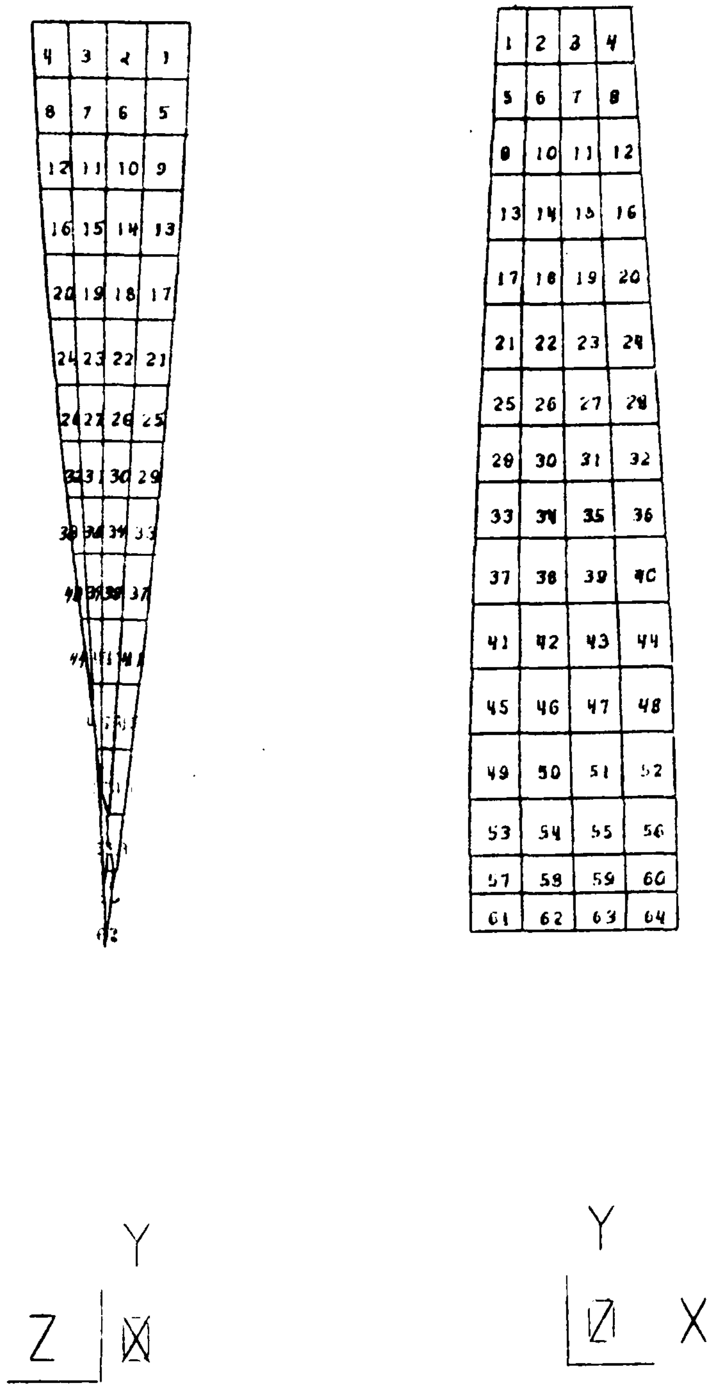


FIGURE 16 TYPICAL BLADE CONFIGURATION

SECTION V
DISCUSSION AND CONCLUSIONS

A quadrilateral composite plate finite element has been added to the SAP IV computer program library to be used on plate type structures. The element is a four-noded sub-parametric flat plate element excluding transverse shear deformations. The element is "incompatible" relative to mid-plane surface rotations along the inter-element boundaries. The element converges relatively well despite the incompatibility, as long as the element maintains a relatively rectangular shape with a reasonable element aspect ratio.

The element was run, modelling various simple plate/beam configurations and showed excellent results. More complex models were designed to test the composite laminate behavior of the element, as described in section IV. The models included flat, cylindrical and doubly-curved shells. The results obtained compared favorably with classical series solutions. (approximately 1-6% disagreement) The results obtained in section 4.9 for the typical blade configuration shows that the NASTRAN and SAP (TYPE 6) element values were slightly lower than the experimental number while the result of the new element (TYPE 9) was slightly higher. The new element is based on an incompatible formulation and in general, convergence is not guaranteed. But, in more cases, the element has been found to be slightly stiffer than compatible elements and therefore the frequency is higher.

The quadrilateral element will be inherently stiff if the four points do not represent a flat surface. Therefore the "quad" element was relaxed to better represent shell behavior by allowing the flat plate element stiffness coefficients to be transformed relative to the local nodal coordinates of the element.

The thin plate theory used to develop the element does not account for normal torsional effects. Therefore, non global adjacent elements, when assembled, will produce a singularity

normal to the plate if the two elements are coplanar. To avoid this singularity in general, an artificial torsional stiffness or scaffolding matrix was added to each of the four nodes. This does violate element equilibrium but, if the magnitude of the coefficients are maintained relatively "soft" compared to the plate bending characteristics, overall equilibrium is closely maintained.

The composite plate element can be used in the existing static and dynamic analyses contained within the SAP IV program. It can be used with all the existing elements in the finite element library as long as the model effects are correct.

The element was not being developed to degenerate to a triangular plate element. If the element is used as triangular, the local effect is "too stiff". If the fourth of the quadrilateral nodes is placed mid-plane on a triangular side, a better approximation can be obtained.

The composite element presently does not contain geometric stiffening effects such as those required in high speed centrifuge machinery.

Further Developments:

Further work should be done to include the geometric stiffening coefficients to account for centripetal acceleration effects of high spin blade systems. This geometric matrix of coefficients would allow a better approximation of blade bending stresses and closer representation of blade natural frequencies at high spin.

A pre-processor program should be developed to handle complex material laminates as a function of blade position. This program would compute the $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{D}$ matrices and thermal vectors needed in the SAP program and produce a storage file for stress recovery. This program should also plot all information for input checking.

A post-processor should be written to retrieve stress information and material information, by lamina, to be used with deformation output and curvatures at the mid-plane surfaces to produce individual lamina stress to be used in failure criteria.

APPENDICES

APPENDIX A - INPUT TO ELEMENT TYPE 9 IN SAP4A

The complete input to the SAP IV program is described in reference 1. The new element TYPE 9 defined in SAP4A is included here and should be appended to reference 1. The input to element TYPE 9 is similar to element TYPE 6. In fact, element TYPE 9 could replace TYPE 6. The format of the input description is consistent with the SAP IV input. Variables labelled integer must be right justified and floating point variables should include a decimal.

COMPOSITE PLATE INPUT TO SAP4A

TYPE 9 Composite Plate Element (QUADRILATERAL)

Note	Columns	Variable	Remark
A.	Control Card (6I5, I10)		
	5	NPAR(1)	Number 9
	6-10	NPAR(2)	Number of Plate Elements
	11-15	NPAR(3)	Number of different materials
(1)	16-20	NPAR(4)	Material Type Key =0 Composite Material Prop. =1 Standard Anisotropic properties (same as TYPE 6)
(2)	21-25	NPAR(5)	Number of Global Material Vectors If zero or blank then global X direction is assumed to be material x axis.
	26-30	NPAR(6)	Integration Order (default set to 2)
(3)	31-40	NPAR(7)	Rotation Stiffness Factor (integer number)

B. Material Property Information

Two types of material can be input to element type 9: general composite material and anisotropic material.

B.1 Composite Material Properties (NPAR(4).EQ.0)

Five cards must be input for every different material. (NPAR(3))

Card 1: (I10, 20X, 4F10.0)

Columns	Variable	Remark
1-10	NN	Material identification number
11-30		Blank
31-40	DEN	Mass density
41-50	AT(1)	
51-60	AT(2)	$\underline{A}_T$ thermal vector components (equation 66)
61-70	AT(3)	

Card 2: (6F10.0)

Columns	Variable	Remark
1-10	BT(1)	
11-20	BT(2)	
21-30	BT(3)	$\underline{B}_T$ thermal vector components (equation 67)
31-40	DT(1)	
41-50	DT(2)	$\underline{D}_T$ thermal vector components (equation 68)
51-60	DT(3)	

Card 3: (6F10.0)

1-10	A(1,1)	
11-20	A(1,2)	$\underset{\sim}{A}$ Matrix coefficients (upper triangular) (equation 63)
21-30	A(1,3)	
31-40	A(2,2)	
41-50	A(2,3)	
51-60	A(3,3)	

Card 4: (6F10.0)

1-10	B(1,1)	
11-20	B(1,2)	$\underset{\sim}{B}$ Matrix coefficients (upper triangular) (equation 64)
21-30	B(1,3)	
31-40	B(2,2)	
41-50	B(2,3)	
51-60	B(3,3)	

Card 5: (6F10.0)

1-10	D(1,1)	
11-20	D(1,2)	$\underset{\sim}{D}$ Matrix coefficients (upper triangular) (equation 65)
21-30	D(1,3)	
31-40	D(2,2)	
41-50	D(2,3)	
51-60	D(3,3)	

3.2 Anisotropic Material Properties (NPAR(4).EQ.1)

(4) Two cards must be input for every different material (NPAR(3))

Card 1: (I10, 20X, 4F10.0)

1-10	NN	Material identification number
11-30		Blank
31-40	DEN	Mass density
41-50	AX	Thermal expansion coefficient αx
51-60	AY	Thermal expansion coefficient αy
61-70	AXY	Thermal expansion coefficient αxy

Card 2: (6F10.0)

1-10	CXX	Elasticity element Cxx
11-20	CXY	Elasticity element Cxy
21-30	CXS	Elasticity element Cxs
31-40	CYY	Elasticity element Cyy
41-50	CYS	Elasticity element Cys
51-60	GXY	Elasticity element Gxy

C. Global Material Vectors (25, 5X, 3F10.0)

NPAR(5) material vectors must be input (except if zero or blank)

1-5	NV	Material vector identification number
6-10		Blank
11-20	DX	X direction cosine
21-30	DY	Y direction cosine
31-40	DZ	Z direction cosine

D. Element Load Multipliers (5 cards)

Card 1: (4F10.0)

1-10	PA	Distributed lateral load multiplier for load case A
11-20	PB	Distributed lateral load multiplier for load case B
21-30	PC	Distributed lateral load multiplier for load case C
31-40	PD	Distributed lateral load multiplier for load case D

Card 2: (4F10.0)

1-10	TA	Temperature multiplier for load case A
11-20	TB	Temperature multiplier for load case B
21-30	TC	Temperature multiplier for load case C
31-40	TD	Temperature multiplier for load case D

Card 3: (4F10.0)

1-10	XA	X-direction acceleration for load case A
11-20	XB	X-direction acceleration for load case B
21-30	XC	X-direction acceleration for load case C
31-40	XD	X-direction acceleration for load case D

Card 4: (4F10.0)

1-10	YA	Y-direction acceleration for load case A
11-20	YB	Y-direction acceleration for load case B
21-30	YC	Y-direction acceleration for load case C
31-40	YD	Y-direction acceleration for load case D

Card 5: (4F10.0)

1-10	ZA	Z-direction acceleration for load case A
11-20	ZB	Z-direction acceleration for load case B
21-30	ZC	Z-direction acceleration for load case C
31-40	ZD	Z-direction acceleration for load case D

E. Element Cards (5I5, I2, I3, I2, I3, I5, 4F10.0)

One card for each NPAR(2) element.

	1-5	NN	Element number
(5)	6-10	I	Node I
	11-15	J	Node J
	16-20	K	Node K
	21-25	L	Node L
(6)	26-27	NG	No. of Gauss integration point.
(7)	28-30	IV	Material vector identification number
(8)	31-32	IREUSE	Previous Element re-use code =0 new element =1 use previous element
(9)	33-35	IM	Material identification number
(10)	36-40	INCL	Element generation parameter
(11)	41-50	TH	Element thickness
(12)	51-60	PR	Element lateral pressure
(13)	61-70	TO	Mean temperature variation from the reference level in undeformed position.
	71-80	TG	Mean temperature gradient across the shell thickness.

Notes:

(1) Element TYPE 9 allows two different form of input. The first form is for laminate matrices while the second is for anisotropic matrices. The later form is identical to element TYPE 6 input.

(2) A material global axis must be defined relative to the material properties formed in the A, B and D matrices.

(3) Rotational Stiffness Factor is set by multiplying NPAR(7) times 1.E-8. Default for NPAR(7) is 100.

(4) Material input in this section is identical to that of element TYPE 6.

(5) The I,J,K and L indices define the element connectivity and also the element normal. The element "z" coordinate is formed by the right hand rule as I goes to J goes to K, etc. The element local axis is determined by the projection of the global material axis onto the element. Once the x-axis is determined, the local y is formed from z and x. All stress output is in this reference. If node L equal K or if zero or left blank, the program will assume that the element is triangular. The resulting local stiffness is then "too stiff".

(6) The number of Gauss integration points can vary as 2 or 3. If a value is set above or below these numbers, the program will reset it to 2. The default value is set NPAR(6).

(7) The global material vector must be greater than or equal to 1 and less than or equal to NPAR(5). If NPAR(5) is blank or zero, then NPAR(5) is set equal to 1 and the global vector is aligned along the global X axis. Default value is set to 1.

(8) If an element has the same planar size, same orientation in space and the same element loading parameters as the previous element, then setting IREUSE equal to 1 will use the same global element stiffness and load vectors for assembly. Default is set to 0.

(9) The material ID number must be between 1 and NPAR(3). Default is set to 1.

(10) Element Generation Parameter: Element cards must be in element number sequence. If element cards are omitted, the program will generate the missing cards as follows:

The increment for the element number is one.

$$\text{i.e.,} \quad NN_{i+1} = NN_i + 1$$

The corresponding increment for nodal connectivity is INCL.

$$\text{i.e.,} \quad I_{i+1} = I_i + \text{INCL}$$

$$J_{i+1} = J_i + \text{INCL}$$

$$K_{i+1} = K_i + \text{INCL}$$

$$L_{i+1} = L_i + \text{INCL}$$

If INCL is left blank then INCL is set to 1. Material identification, element thickness, distributed lateral load, temperature and temperature gradient for the element are then the same as for the first element in the generated group. The last element card must be input to exit element group properly.

(11) The plate thickness is used mainly to compute the mass of the element. Default is set to 1.0.

(12) The pressure is normal to the surface of the element. The positive direction for the pressure loading vector is in the positive direction of the local z coordinate.

(13) The temperature required is the mean temperature difference (TO) from the reference temperature of the element in a undeformed state. The TG is the mean thermal gradient through the element thickness.

APPENDIX B - INPUT TO PRE-PROCESSOR PROGRAM LAYUP

A pre-processor program was developed to calculate the $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{D}$ matrices as described in equations 63 through 65 and the thermal load vectors described in equations 66 through 68. The material information by layer is input be lamina position and fiber orientation. The pre-processor then computes the $\underset{\sim}{\tilde{C}}$ matrices involving the tensor transformation and constructs the element material matrix $\underset{\sim}{E}_m$ relative to the mid-plane of the plate. Similar thermal vectors are also computed and output. This information is then used directly for SAP4A - TYPE 9.

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INPUT TO PROGRAM LAYUP

A. Number of Cases Card (I2)

Note	Columns	Variable	Remark
(1)	1-2	NCASES	Enter the total number of laminate configurations to be considered.

B. Heading or Title Card (8A10)

Note	Columns	Variable	Remark
(2)	1-80	ITITLE	Enter the title information to be printed with the output.

C. Laminate Data Card (4I5)

Note	Columns	Variable	Remark
(3)	1-5	NLAM	Enter the number of laminae in the laminate.
(4)	6-10	NMAT	Enter the number of different materials in the laminate.
	11-15	ITRAN	If ITRAN is zero or blank the transformed thermal properties of each lamina will not be part of the output.
	16-20	IORD	If IORD is left blank the laminate ordinates will not be output.

D. Material Property Card (6F10.3)

Note	Columns	Variable	Remark
(5)	1-10	E11(K)	Enter lamina modulus in fiber direction
	11-20	E22(K)	Enter lamina modulus in direction transverse to fibers.
	21-30	XN1(K)	Enter major Poisson's ratio.
	31-40	G12(K)	Enter lamina shear modulus.
	41-50	THERM1(K)	Enter C.T.E. in fiber direction.
	51-60	THERM2(K)	Enter C.T.E. in transverse direction.

E. Lamina Data Card (F10.3, I10, F10.3)

Note	Columns	Variable	Remark
	1-10	T(J)	Enter thickness of the Jth lamina.
	11-20	MATL(J)	Enter the number that identifies the material of the Jth lamina.
	21-30	PJI(J)	Enter the orientation of the Jth lamina with respect to the laminate axes.

Notes:

(1) There are no program restrictions on the number of cases that may be analyzed in a single run.

(2) Begin each new data case with a heading card.

(3) The program is currently capable of handling up to 48 laminae per layup. This can be increased by changing the appropriate dimension statements as shown in the program listing.

(4) Although no limitation on the number of different materials need be imposed, the program dimension statements currently allow for a maximum of NMAT = 4. However, this can also be increased if necessary.

(5) This material data is vendor information. One card is required for each material (i.e., K = 1, NMAT).

(6) One card is required, per layer, in the laminate. (i.e., J = 1, NLAM).

APPENDIX C - COMPUTER RUN - INPUT AND OUTPUT

This appendix contains the complete input and resulting output of the SAP4A program using the new composite plate element (TYPE 9). The example is the model contained in section 4.7 (Curved Plate under Uniform Pressure). The first section contains a listing of the card images used to execute the program. The second section is the complete output resulting from this input.

SAP4A

SAP4A

FOR THE STATIC AND DYNAMIC
ANALYSIS OF LINEAR SYSTEMS
USING FINITE ELEMENTS.

VERSION 4A DEVELOPED AT THE
UNIVERSITY OF LOWELL, LOWELL
MASS 01854 JUNE, 1979

SAP 4A TEST CASE CURVED PLATE (0/90/90/0)

C O N T R O L   I N F O R M A T I O N

```
NUMBER OF NODAL POINTS    =  25
NUMBER OF ELEMENT TYPES   =   1
NUMBER OF LOAD CASES      =   1
NUMBER OF FREQUENCIES     =   0
ANALYSIS CODE (NDYN)      =   0
   EQ.0,  STATIC
   EQ.1,  MODAL EXTRACTION
   EQ.2,  FORCED RESPONSE
   EQ.3,  RESPONSE SPECTRUM
   EQ.4,  DIRECT INTEGRATION
SOLUTION MODE (MODEX)     =   0
   EQ.0,  EXECUTION
   EQ.1,  DATA CHECK
NUMBER OF SUBSPACE
INTERATION VECTORS (NAD) =    0
EQUATIONS PER BLOCK       =   0
TAPE10 SAVE FLAGE (N10SV)=    0
```

NODAL POINT INPUT DATA

NODE NUMBER	BOUNDARY CONDITION CODES X	Y	Z	XX	YY	ZZ	NODAL POINT COORDINATES X	Y	Z		T
1	-1	1	-1	1	-1	0	0.000	0.000	0.000	0	0.000
2	0	0	0	0	0	0	1.250	0.000	.255	0	0.000
3	0	0	0	0	0	0	2.500	0.000	.443	0	0.000
4	0	0	0	0	0	0	3.750	0.000	.557	0	0.000
5	1	0	0	0	1	1	5.000	0.000	.597	0	0.000
6	0	1	1	1	0	0	0.000	1.250	0.000	0	0.000
7	0	0	0	0	0	0	1.250	1.250	.255	0	0.000
8	0	0	0	0	0	0	2.500	1.250	.443	0	0.000
9	0	0	0	0	0	0	3.750	1.250	.557	0	0.000
10	1	0	0	0	1	1	5.000	1.250	.597	0	0.000
11	0	1	1	1	0	0	0.000	2.500	0.000	0	0.000
12	0	0	0	0	0	0	1.250	2.500	.255	0	0.000
13	0	0	0	0	0	0	2.500	2.500	.443	0	0.000
14	0	0	0	0	0	0	3.750	2.500	.557	0	0.000
15	1	0	0	0	1	1	5.000	2.500	.597	0	0.000
16	0	1	1	1	0	0	0.000	3.750	0.000	0	0.000
17	0	0	0	0	0	0	1.250	3.750	.255	0	0.000
18	0	0	0	0	0	0	2.500	3.750	.443	0	0.000

19	0	0	0	0	0	0	3.750	3.750	.557	0	0.000
20	1	0	0	0	1	1	5.000	3.750	.597	0	0.000
21	0	-1	1	-1	0	0	0.000	5.000	0.000	0	0.000
22	0	0	0	0	0	0	1.250	5.000	.255	0	0.000
23	0	0	0	0	0	0	2.500	5.000	.443	0	0.000
24	0	0	0	0	0	0	3.750	5.000	.557	0	0.000
25	1	1	0	1	1	1	5.000	5.000	.597	0	0.000

GENERATED NODAL DATA

NODE	BOUNDARY CONDITION CODES						NODAL POINT COORDINATES			
NUMBER	X	Y	Z	XX	YY	ZZ	X	Y	Z	T
1	-1	1	-1	1	-1	0	0.000	0.000	0.000	0.000
2	-1	0	-1	0	-1	0	1.250	0.000	.255	0.000
3	-1	0	-1	0	-1	0	2.500	0.000	.443	0.000
4	-1	0	-1	0	-1	0	3.750	0.000	.557	0.000
5	1	0	-1	0	1	1	5.000	0.000	.597	0.000
6	0	1	1	1	0	0	0.000	1.250	0.000	0.000
7	0	0	0	0	0	0	1.250	1.250	.255	0.000
8	0	0	0	0	0	0	2.500	1.250	.443	0.000
9	0	0	0	0	0	0	3.750	1.250	.557	0.000
10	1	0	0	0	1	1	5.000	1.250	.597	0.000
11	0	1	1	1	0	0	0.000	2.500	0.000	0.000
12	0	0	0	0	0	0	1.250	2.500	.255	0.000
13	0	0	0	0	0	0	2.500	2.500	.443	0.000
14	0	0	0	0	0	0	3.750	2.500	.557	0.000
15	1	0	0	0	1	1	5.000	2.500	.597	0.000
16	0	1	1	1	0	0	0.000	3.750	0.000	0.000
17	0	0	0	0	0	0	1.250	3.750	.255	0.000
18	0	0	0	0	0	0	2.500	3.750	.443	0.000
19	0	0	0	0	0	0	3.750	3.750	.557	0.000
20	1	0	0	0	1	1	5.000	3.750	.597	0.000
21	0	-1	1	-1	0	0	0.000	5.000	0.000	0.000
22	0	-1	0	-1	0	0	1.250	5.000	.255	0.000
23	0	-1	0	-1	0	0	2.500	5.000	.443	0.000
24	0	-1	0	-1	0	0	3.750	5.000	.557	0.000
25	1	1	0	1	1	1	5.000	5.000	.597	0.000

EQUATION NUMBERS

N	X	Y	Z	XX	YY	ZZ
1	0	0	0	0	0	1
2	0	2	0	3	0	4
3	0	5	0	6	0	7
4	0	8	0	9	0	10
5	0	11	0	12	0	0
6	13	0	0	0	14	15
7	16	17	18	19	20	21
8	22	23	24	25	26	27
9	28	29	30	31	32	33
10	0	34	35	36	0	0
11	37	0	0	0	38	39
12	40	41	42	43	44	45
13	46	47	48	49	50	51
14	52	53	54	55	56	57
15	0	58	59	60	0	0
16	61	0	0	0	62	63
17	64	65	66	67	68	69
18	70	71	72	73	74	75
19	76	77	78	79	80	81
20	0	82	83	84	0	0
21	85	0	0	0	86	87
22	88	0	89	0	90	91
23	92	0	93	0	94	95
24	96	0	97	0	98	99
25	0	0	100	0	0	0

C O M P O S I T E

```
ELEMENT TYPE              =    9
NUMBER OF ELEMENTS        =   16
NUMBER OF MATERIALS       =    1
MATERIAL TYPE KEY         =    0
   = 0, COMPOSITE MAT
   = 1, ANISOTROPIC MAT
NO. OF MATERIAL VECT      =    0
INTEGRATION ORDER (2)     =    2
ROTATIONAL STIF FACT      =  .00000100
```

COMPOSITE MATERIAL PROPERTY TABLE (ABD)

MATERIAL NUMBER	MASS DENSITY	A B D M A T R I X C O E F F I C I E N T S					
		AT(1)	AT(2)	AT(3)			
		BT(1)	BT(2)	BT(3)	DT(1)	DT(2)	DT(3)
		A(1,1)	A(1,2)	A(1,3)	A(2,2)	A(2,3)	A(3,3)
		B(1,1)	B(1,2)	B(1,3)	B(2,2)	B(2,3)	B(3,3)
		D(1,1)	D(1,2)	D(1,3)	D(2,2)	D(2,3)	D(3,3)
1	0.	0.	0.	0.			
		0.	0.	0.	0.	0.	0.
		.333E+07	.200E+06	0.	.333E+07	0.	.200E+06
		0.	0.	0.	0.	0.	0.
		.178E+07	.667E+03	0.	.425E+04	0.	.667E+03

ELEMENT LOAD CASE MULTIPLIERS

ELEMENT LOAD CASE NUMBER	PRESSURE	THERMAL EFFECTS	X-ACCELERATION	Y-ACCELERATION	Z-ACCELERATION
1	1.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000

THIN COMPOSITE PLATE ELEMENT DATA

ELEMENT NUMBER	NODE-I	NODE-J	NODE-K	NODE-L	INTEG POINTS	GLOBAL VECTOR	REUSE CODE	MAT NUMBER	INCRE INDEX	AVERAGE THICKNESS	NORMAL PRESSURE	TEMPERATURE DIFFERENCE	THE[illegible] GRADIE[illegible]	
1	1	2	7	6	2	1	0	1	1	.2000	1.0	1.00	0.[illegible]	[illegible]
2	2	3	8	7	2	1	0	1	1	.200	1.0	1.00	0.[illegible]	[illegible]
3	3	4	9	8	2	1	0	1	1	.200	1.0	1.00	0.00[illegible]	[illegible]
4	4	5	10	9	2	1	0	1	1	.2000	1.0	1.00	0.00[illegible]	[illegible]
5	6	7	12	11	2	1	0	1	1	.2000	1.0	1.00	0.00[illegible]	[illegible]
6	7	8	13	12	2	1	0	1	1	.2000	1.0	1.00	0.00[illegible]	3[illegible]
7	8	9	14	13	2	1	0	1	1	.2000	1.0	1.00	0.000	[illegible]
8	9	10	15	14	2	1	0	1	1	.2000	1.0	1.00	0.00[illegible]	[illegible]
9	11	12	17	16	2	1	0	1	1	.2000	1.0	1.00	0.000	33
10	12	13	18	17	2	1	0	1	1	.2000	1.0	1.00	0.000	[illegible]
11	12	13	18	17	2	1	0	1	1	.2000	1.0	1.00	0.000	[illegible]
12	13	14	19	18	2	1	0	1	1	.2000	1.0	1.00	0.000	36
13	16	17	22	21	2	1	0	1	1	.2000	1.0	1.00	0.000	31
14	17	18	23	22	2	1	0	1	1	.2000	1.0	1.00	0.000	32
15	18	19	24	23	2	1	0	1	1	.2000	1.0	1.00	0.000	30
16	19	20	25	24	2	1	0	1	1	.2000	1.0	1.00	0.000	25

E Q U A T I O N   P A R A M E T E R S

TOTAL NUMBER OF EQUATIONS = 100
BANDWIDTH = 36
NUMBER OF EQUATIONS IS A BLOCK = 100
NUMBER OF BLOCKS = 1
WORKING STORAGE SIZE (MTDT) =15000

N O D A L   L O A D S   (S T A T I C)   O R   M A S S E S   (D Y N A M I C)

NODAL NUMBER	LOAD CASE	X-AXIS FORCE	Y-AXIS FORCE	Z-AXIS FORCE	X-AXIS MOMENT	Y-AXIS MOMENT	Z-AXIS MOMENT

STRUCTURE LOAD CASE	ELEMENT LOAD MULTIPLIERS A	B	C	D
1	1.000	0.000	0.000	0.000

```
*********************************************************
* ENTERING SOLUTION OF EQUATIONS, CP  TIME  =    6.546 *
*********************************************************


*****************************************************************
*  1 BLOCK OF EQUATIONS HAS BEEN REDUCED, CP  TIME  =    7.301 *
*****************************************************************


******************************************************************************
* START OF BACK SUBSTITUTION FOR DISPLACEMENT VECTORS, CP  TIME  =    7.323 *
******************************************************************************


******************************************************************************
*  END OF BACK SUBSTITUTION FOR DISPLACEMENT VECTORS, CP  TIME  =    7.557 *
******************************************************************************
```

NODE DISPLACEMENTS/ROTATIONS

NODE NUMBER	LOAD CASE	X- TRANSLATION	Y- TRANSLATION	Z- TRANSLATION	X- ROTATION	Y- ROTATION	Z- ROTATION
25	1	0.	0.	.23686E-02	0.	0.	0.
24	1	-.43330E-05	0.	.22341E-02	0.	-.22846E-03	.24251E-02
23	1	.29910E-04	0.	.17740E-02	0.	-.50339E-03	.12619E-02
22	1	.14209E-03	0.	.99125E-03	0.	-.73114E-03	.55742E-03
21	1	.34210E-03	0.	0.	0.	-.82308E-03	-.33287E-04
20	1	0.	-.74597E-05	.27501E-03	.27501E-03	0.	0.
19	1	-.20380E-06	-.65600E-05	.20153E-02	.27783E-02	-.25064E-03	-.56320E-03
18	1	.36523E-04	-.49517E-05	.15643E-02	.21248E-03	-.46662E-03	-.14188E-03
17	1	.13881E-03	-.24203E-05	.86496E-03	.11450E-03	-.64243E-03	-.36043E-04
16	1	.31413E-03	0.	0.	0.	-.71667E-03	-.46018E-04
15	1	0.	-.13913E-04	.17051E-02	.48202E-03	0.	0.
14	1	-.19080E-05	-.12181E-04	.15900E-02	.43645E-03	-.18513E-03	.93371E-04
13	1	.24801E-04	-.91584E-05	.12439E-02	.33353E-03	-.36472E-03	-.12188E-04
12	1	.10455E-03	-.44811E-05	.69050E-03	.17260E-03	-.51126E-03	.40137E-06
11	1	.24417E-03	0.	0.	0.	-.57267E-03	-.78111E-04
10	1	0.	-.18200E-04	.96545E-03	.70068E-03	0.	0.
9	1	-.25439E-05	-.15923E-04	.90334E-03	.65577E-03	-.99547E-04	-.29545E-04
8	1	.10698E-04	-.11987E-04	.71517E-03	.51507E-03	-.20118E-03	-.34591E-04
7	1	.54882E-04	-.58601E-05	.40317E-03	.29269E-03	-.29423E-03	-.90744E-04
6	1	.13654E-03	0.	0.	0.	-.33039E-03	-.93899E-04
5	1	0.	-.19701E-04	0.	.80782E-03	0.	0.
4	1	0.	-.17229E-04	0.	.75727E-03	0.	-.62768E-05
3	1	0.	-.12988E-04	0.	.60703E-03	0.	-.53770E-04
2	1	0.	-.63498E-05	0.	.35831E-03	0.	-.14308E-03
1	1	0.	0.	0.	0.	0.	-.95865E-04

[illegible]

NUMBER	CASE	MEMBRANE STRESS COMPONENTS / MIDPLANE STRAIN COMPONENTS SXX EXX	SYY EYY	SXY EXY	BENDING MOMENT COMPONENTS / BENDING MOMENT CURVATURES MXX KXX	MYY KYY	MXY KXY
1	1	.1443E+01	.75[illegible]E+00	.2046E+02	.3076E+00	.1019E+00	-.3552E+00
1	1	.4[illegible]05E-06	.1959E-06	.1023E-03	.1652E-04	.2138E-04	-.5328E-03
2	1	.3693E+01	.2202E+01	.1743E+02	.6920E+00	.2714E+00	-.2711E+00
2	1	.1072E-05	.5963E-06	.8742E-04	.3680E-04	.5808E-04	-.4067E-03
3	1	.5366E+01	.3387E+01	.1176E+02	.7701E+00	.3552E+00	-.1621E+00
3	1	.1555E-05	.9227E-06	.5879E-04	.4049E-04	.7722E-04	-.2432E-03
4	1	.6260E+01	.4104E+01	.4198E+01	.7622E+00	.3823E+00	-.5348E-01
4	1	.1811E-05	.1123E-05	.2099E-04	.3980E-04	.8371E-04	-.8023E-04
5	1	.3218E+01	.2025E+01	.1616E+02	.7462E+00	.1907E+00	-.2460E+00
5	1	.9323E-06	.5516E-06	.8081E-04	.4059E-04	.3852E-04	-.3690E-03
6	1	.8315E+01	.5088E+01	.1391E+02	.1757E+01	.5415E+00	-.2086E+00
6	1	.2394E-05	.1683E-05	.6953E-04	.9477E-04	.1126E-03	-.3129E-03
7	1	.1213E+02	.9456E+01	.9450E+01	.2091E+01	.7309E+00	-.1359E+00
7	1	.3481E-05	.2628E-05	.4725E-04	.1120E-03	.1544E-03	-.2038E-03
8	1	.1413E+02	.1151E+02	.3380E+01	.2136E+01	.8134E+00	-.4409E-01
8	1	.4046E-05	.3211E-05	.1690E-04	.1138E-03	.1735E-03	-.6613E-04
9	1	.3323E+01	.2937E+01	.1041E+02	.9590E+00	.1334E+00	-.1464E+00
9	1	.9475E-06	.8243E-06	.5205E-04	.5316E-04	.2304E-04	-.2196E-03
10	1	.8677E+01	.8847E+01	.8948E+01	.2317E+01	.4282E+00	-.1242E+00
10	1	.2453E-05	.2507E-05	.4474E-04	.1275E+03	.8075E-04	-.1864E-03
11	1	.1304E+02	.1384E+02	.6120E+01	.2890E+01	.6999E+00	-.9760E-01
11	1	.3675E-05	.3931E-05	.3060E-04	.1576E-03	.1400E-03	-.1464E-03

12	1	.1558E+02	.1698E+02	.2221E+01	.3195E+01	.7731E+00	-.4641E-01
12	1	.4383E-05	.4830E-05	.1111E-04	.1742E-03	.1546E-03	-.6962E-04
13	1	.4471E+01	.3484E+01	.4280E+01	.1154E+01	.2850E-01	-.1025E+00
13	1	.1283E-05	.9681E-06	.2140E-04	.6513E-04	-.3511E-05	-.1538E-03
14	1	.1159E+02	.1049E+02	.3534E+01	.2840E+01	.1511E+00	-.5924E-01
14	1	.3300E-05	.2999E-05	.2707E-04	.1570E-05	.2077E-04	-.2034E-05
15	1	.1695E+02	.1631E+02	.2129E+01	.3495E+01	.2823E+00	.2421E-01
15	1	.4807E-05	.4605E-05	.1065E-04	.1955E-03	.3575E-04	.3631E-04
16	1	.1979E+02	.1981E+02	.6479E+00	.3522E+01	.9045E+00	.2385E-01
16	1	.5600E-05	.5603E-05	.3240E-05	.1915E-03	.1828E-03	.3578E-04

S T A T I C   S O L U T I O N   T I M E   L O G

```
EQUATION SOLUTION    =   1.04
DISPLACEMENT OUTPUT  =    .14
STRESSS RECOVERY     =    .32
```

O V E R A L L   T I M E   L O G

```
NODAL POINT INPUT              =    .43
ELEMENT STIFFNESS FORMATION    =   5.12
NODAL LOAD INPUT               =    .05
TOTAL STIFFNESS FORMATION      =    .46
STATIC ANALYSIS                =   1.51
EIGENVALUE EXTRACTION          =   0.00
FORCED RESPONSE ANALYSIS       =   0.00
RESPONSE SPECTRUM ANALYSIS     =   0.00
STEP-BY-STEP INTEGRATION       =   0.00

TOTAL SOLUTION TIME            -   7.57
```

```
SAP 4A TEST CASE CURVED PLATE (0/90/90/0)
   25    1    1
    1   -1    1   -1    1   -1     0.        0.        0.
    2                              1.25      0.         .255
    3                              2.5       0.         .443
    4                              3.75      0.         .557
    5    1                    1   15.0       0.         .597
    6         1    1    1          0.        1.25      0.
    7                              1.25      1.25       .255
    8                              2.5       1.25       .443
    9                              3.75      1.25       .557
   10    1                    1   15.0       1.25       .597
   11         1    1    1          0.        2.5       0.
   12                              1.25      2.5        .255
   13                              2.5       2.5        .443
   14                              3.75      2.5        .557
   15    1                    1   15.0       2.5        .597
   16         1    1    1          0.        3.75      0.
   17                              1.25      3.75       .255
   18                              2.5       3.75       .443
   19                              3.75      3.75       .557
   20    1                    1   15.0       3.75       .597
   21        -1   -1   -1          0.        5.0       0.
   22                              1.25      5.0        .255
   23                              2.5       5.0        .433
   24                              3.75      5.0        .557
   25    1    1         1    1    15.0       5.0        .597
    9   16    1
         1

3.333333+6.2      +6          3.333333+6         .2       +6

17.75    +3 .666666+3         4.25    +3         .6666666+3
```

1	1	2	7	6	.2	1.	1.
2	2	3	8	7	.2	1.	1.
3	3	4	9	8	.2	1.	1.
4	4	5	10	9	.2	1.	1.
5	6	7	12	11	.2	1.	1.
6	7	8	13	12	.2	1.	1.
7	8	9	14	13	.2	1.	1.
8	9	10	15	14	.2	1.	1.
9	11	12	17	16	.2	1.	1.
10	12	13	18	17	.2	1.	1.
11	13	14	19	18	.2	1.	1.
12	14	15	20	19	.2	1.	1.
13	16	17	22	21	.2	1.	1.
14	17	18	23	22	.2	1.	1.
15	18	19	24	23	.2	1.	1.
16	19	20	25	24	.2	1.	1.

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